THE BOSTON COLLOQUIUM OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Fourth Colloquium * of the AMERICAN MATHEMATICAL SOCIETY was held, in connection with the tenth summer meeting, at the Massachusetts Institute of Technology, Boston, Mass., opening on Wednesday morning, September 2, and closing on the following Saturday morning. The total attendance at the several lectures was gratifyingly large, including the following thirty-one members of the Society and other persons interested in mathematics:

Professor F. H. Bailey, Dr. G. A. Bliss, Dr. C. L. Bouton, Professor Ellen L. Burrell, Professor F. N. Cole, Professor L. L. Conant, Professor E. W. Davis, Mr. H. N. Davis, Professor W. P. Durfee, Mr. W. B. Ford, Mr. N. R. George, Jr., Miss A. B. Gould, Rev. J. G. Hagen, Dr. C. N. Haskins, Dr. Gustavus Hay, Professor J. I. Hutchinson, Dr. Edward Kasner, Professor H. P. Manning, Dr. C. M. Mason, Mr. E. A. Miller, Professor Frank Morley, Professor W. F. Osgood, Dr. A. B. Pierce, Professor I. J. Schwatt, Miss M. E. Trueblood, Professor H. W. Tyler, Professor E. B. Van Vleck, Dr. Roxana H. Vivian, Professor H. S. White, Professor F. S. Woods, Professor T. W. D. Worthen.

Three courses of lectures were given, as follows:

Professor E. B. VAN VLECK: "Selected topics in the theory of divergent series and of continued fractions." Six lectures.

Professor H. S. WHITE: "Linear systems of curves on algebraic surfaces." Three lectures.

Professor F. S. WOODS: "The connectivity of non-euclidean space." Three lectures.

Two lectures were given each morning, and two on each of the afternoons of Wednesday and Friday, the courses alternating with one another. Thursday afternoon, which was thus left free, was devoted to an excursion to Nantasket, in Boston harbor.

A synopsis of Professor White's lectures follows in the

* Interesting data concerning the previous colloquia are given in Dr. Edward Kasner's report of the Ithaca Colloquium, BULLETIN, Vol. 8 (1901), pp. 22–25.

present number of the BULLETIN. Reports of the lectures of Professors Van Vleck and Woods will appear in later numbers.

F. N. COLE.

COLUMBIA UNIVERSITY.

LINEAR SYSTEMS OF CURVES UPON ALGEBRAIC SURFACES.

AN ABSTRACT OF THREE LECTURES DELIVERED AT THE BOSTON COLLOQUIUM, SEPTEMBER 2-5, 1903.

BY PROFESSOR H. S. WHITE.

THE notion of equivalence as formulated in projective geometry has simplified greatly the study of algebraic curves and surfaces, particularly those of low order. The next step toward a wider survey is the admission of all birational transformations of the plane, or of space of three or more dimensions. In the plane, the theory of Cremona transformations is no longer new, and the elements are familiar to all students of geometry. Not so, however, in space of more than two dimensions; probably for the reason that nothing is known analogous to the theorem that a plane Cremona transformation is resolvable into a succession of quadric transformations and collineations. And even in plane geometry the intricacies of the transformations themselves have kept most students from the matter of higher importance, the properties of figures that remain invariant under all transformations of the group. Yet there does exist a body of doctrine under the accepted title of "Geometry upon an algebraic curve," and a fair beginning has been made upon a similar theory, the "Geometry upon an algebraic surface."* These titles are intended to cover only such properties of a curve or surface as appertain to the entire class of curves or surfaces that can be related birationally to the fundamental form.

* Consult, for an outline of the geometry upon an algebraic curve, Pascal's Repertorium der höheren Mathematik, Part II, Chapter V, § 4; or the more extended articles: C. Segre, "Introduzione alla geometria sopra un ente algebrico semplicemente infinito"; E. Bertini, "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico,"—both in *Annali di Matematica*, series II, vol. 22 (1894). For the corresponding theories regarding surfaces, the best reference is to the comprehensive summary by Castelnuovo and Enriques: "Sur quelques récents résultats dans la théorie des surfaces algébriques," *Math. Annalen*, vol. 48 (1896).

A plane algebraic curve may have its order changed by a Cremona transformation, but not its deficiency (Genre, Geschlecht). As to sets of points on the curve, two sets which together make up a complete intersection of a second curve with the first do not lose that property by birational transformation, if we exclude from consideration fundamental points introduced by the transformation itself. * Mutually residual set of points, and corresidual sets, preserve their relation. Hence the group of sets of points corresidual with any given set becomes of importance. If a given set of D points lies on a curve of deficiency p, and if a corresidual set can be found containing k arbitrary points, then these numbers are connected by the relation constituting the Riemann-Roch theorem

$$k = D - p + \rho,$$

where ρ is zero if $D > 2p - 2$.

The totality of all sets of D points corresidual to any one set is termed a group or *series*, and is denoted by a symbol g_D^k. Such a series is called *complete*. If by any algebraic restrictions a series is separated out from it, of course that would be called incomplete or partial. For example, on a plane nodal cubic a series g_3^2 is cut out by all straight lines, incomplete because any three arbitrary points of the curve are corresidual to any other three. Every series g_D^k can be cut out upon the fundamental curve by a linear system of auxiliary curves whose equation may be written, with k parameters:

$$F_0 + l_1F_1 + l_2F_2 + \cdots + l_kF_k = 0.$$

As on a single curve sets of points, so in a plane, linear systems of curves are studied. By every birational transformation, linear systems are carried over into linear systems. A complete linear system is defined most easily by specifying the multiplicity that a curve of the system must have in each point of a fundamental set, and by prescribing the *order* of the curves. Thus $\binom{a_1\, a_2 \cdots}{s_1\, s_2 \cdots}$ can indicate that in a_1 every curve is to have a multiple point of order at least s_1, etc. If the base points alone, with their respective multiplicities, determine a system under consideration, that system is termed *complete*. If the base points actually impose, for curves of order m, fewer conditions than would be expected from their several multiplic-

* Or if we employ no auxiliary curves except such as are adjoint to that containing the point sets.

ities, the system is *special;* otherwise it is *regular*. It is an important theorem that no set of r base points can be so *located* as to produce an $(r+1)$th variable multiple point on the curves of the system; *i. e.*, the multiple points of the generic curve of a plane linear system lie all in the base points of the system.

Adjoint curves of a linear system are familiar to the student of function theory; they have in every multiple point of order s for the given system a multiplicity of order at least $s-1$. The adjoints of order lower by 3 than the original system are important from the fact that they transform always into the corresponding system of adjoints to the transformed curves. On this account the term adjoint, as used ordinarily, implies a curve of order $m-3$ unless differently specified. *Second adjoints* are adjoint to adjoints of the system, etc. The employment of successive adjoint systems as a means of investigation is due to S. Kantor and to G. Castelnuovo, the latter acknowledging the priority of the former. * On every curve its adjoints cut out a unique complete series g^{p-1}_{2p-2}, called the *canonical series*. The deficiency of the first or second adjoints of a linear system is denoted by P_1 or P_2, and may be termed first, or second, canonical deficiency. Aside from the canonical series upon curves of a system, the most important are the *characteristic series* of the system, that is the totality of sets of points in which two curves of the system intersect. If a plane linear system is complete, then the *characteristic series* on each curve is a *complete series* upon that curve. So far the definitions and propositions refer to curves in a plane; the question is in order, whether they can be transferred to systems of curves lying upon curved surfaces.

First, it is noticed that by means of a linear system of curves the plane may be related point for point to a surface in space of three or more dimensions.† If the system is k-fold infinite, $k+2$ members of the system can be related arbitrarily to $k+2$ hyperplanes in space of k dimensions. Take $k=3$ for ease; then a curve of the system

$$u_1 f_1 + u_2 f_2 + u_3 f_3 + u_4 f_4 = 0$$

may be assigned to a plane $(u_1 : u_2 : u_3 : u_4)$ in ordinary space.

* See *Math. Annalen*, vol. 44, p. 127.

† Exceptional cases are discussed by Enriques: "Ricerche di geometria sulle superficie algebriche," *Torino Memorie*, series II, vol. 44 (1893), p. 178.

Curves through one point become then planes through one point, and the ∞^2 points of the plane become the ∞^2 points of some algebraic surface F. All such surfaces are called *rational*. Similarly a linear family of curves triply infinite upon any surface relate that surface point for point to another surface in threefold space, linear systems of curves in one giving rise to linear systems upon the other, and *the transformed system will lack fundamental or base points.* The value of such projectively related pictures of a linear system of curves was first emphasized by C. Segre.

If on any surface, rational or not, there exists a system of curves doubly infinite, such that two arbitrary points determine one and only one curve containing them, that may be termed a linear system upon the surface in question; and Enriques proves that the ∞^2 curves of such a system can be *projectively* related to the straight lines of a plane. If the series is ∞^3, then its curves are referable projectively to the planes of three-space, etc. Only *simply* infinite systems escape this far-reaching theorem, and thus give rise to a most interesting unsettled question, indicated by Castelnuovo.*

Definitions of residual and corresidual curves upon a surface are those which any one could formulate at once from the use of these terms for sets of points upon a curve; their significance upon a twisted curve is the same as upon its plane projection. So of complete systems, both of curves and of surfaces, the latter admitting of course multiple curves as well as base points. For a surface of order m, the adjoints invariantively related are of order $m-4$, containing as $(s-1)$-fold curve every s-fold curve of the given surface. If these first adjoint surfaces form a k-fold infinite linear system, the number k is an invariant of the surface, and is termed its *geometric deficiency* (p_g). Attempting to express this number in terms of the order m of the surface, the order d and deficiency π of its double curve (if any), and of the number t of triple points on this double curve, one would find a second number

$$p_n = \tfrac{1}{6}(m-1)(m-2)(m-3) - d(m-4) + 2t + \pi - 1,$$

called the *numerical deficiency* of the surface. This number also is an invariant of the surface, as Noether first proved, and may

* Castelnuovo: "Alcuni risultati sui sistemi lineari di curve appartenenti ad una superficie algebrica." *Memorie di matematica e di fisica della Società Italiana delle Scienze*, Series 3°, vol. 10 (1896), pp. 82–102. See especially the close of his preface.

be *either equal to or less than* p_g, but never greater. Rational surfaces have $p_g = p_a = 0$; ruled surfaces have p_a negative. If $p_g = p_a$, then the above mentioned theorem of Enriques concerning linearity holds true also for systems which are only simply infinite. Surfaces of the first adjoint system cut out upon a given surface a system of curves, each of deficiency $p^{(1)}$ or less. This invariant number $p^{(1)}$ we may call the *canonical deficiency* of the surface; the curves form an unique complete linear system, just as do the point sets of the canonical series on a plane curve.

The definitions here given are but a part of those found useful in this fascinating branch of geometry. The true way to learn something of the subject is not to master first all its definitions and distinctions, but to study the proofs of some few leading theorems. Such are Enriques's proof of the equivalence of two geometrical definitions of the linearity of a system (mentioned above), and the following less elementary propositions:

1. Surfaces whose plane or hyperplane sections are irreducible unicursal curves are either ruled or rational (Noether).*

2. So also surfaces whose plane or hyperplane sections are irreducible elliptic curves (Castelnuovo),† or hyperelliptic of any deficiency π (Enriques).‡ For plane sections, not hyperelliptic, of deficiency $\pi > 2$, the corresponding theorem is not yet fully known.

3. Upon any algebraic surface $f(x, y, z, t) = 0$ a linear differential of first kind is said to exist (Picard), if an expression

$$\int [P_1(x, y, z, t) \cdot dx + P_2 \cdot dy + P_3 \cdot dz + P_4 \cdot dt]$$

is finite and determinate, independent of the path of integration, when taken upon the surface between any two arbitrary points. If the surface $f = 0$ is a *cone*, such differentials exist, for they are the abelian differentials of first kind upon its plane sections. Picard proves§ that if the surface $f = 0$ have no multiple points or curves, then no such differential can exist upon it. The proof of this theorem concluded these lectures.

EVANSTON, ILL.,
September, 1903.

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* Noether's theorem is more general. See *Math. Annalen*, vol. 3: "Ueber Flächen, welche Schaaren rationaler Curven besitzen."

† "Sulle superficie algebriche," etc., *Lincei Rendiconti*, January, 1894.

‡ "Sui sistemi lineari," etc., *Math. Annalen*, vol. 46 (1895), pp. 179–199.

§ Picard et Simart: Théorie des fonctions algébriques de deux variables indépendantes, vol. 1 (1897), pp. 119–120.

AN EXPRESSION OF CERTAIN KNOWN FUNCTIONS AS GENERALIZED HYPERGEOMETRIC FUNCTIONS.

BY MR. E. T. WHITTAKER.

(Read before the American Mathematical Society, August 31, 1903.)

§ 1. *Introduction.*

It has long been known that many functions (*e. g.*, the logarithm and the Legendre functions) are particular cases of the hypergeometric function, and that other functions (*e. g.*, the exponential function and the Bessel functions) are limiting cases of the hypergeometric function. The object of the present paper is to show that the latter class of functions is much larger than has hitherto been supposed; that, in fact, it includes (in addition to those functions already known to belong to it) the following five types of functions, namely:

1. The functions which arise in harmonic analysis in connection with the parabolic cylinder.

2. The error function, which arises in connection with the theories of probability, errors of observations, refraction, and conduction of heat.

3. The incomplete gamma function, studied by Legendre, Hočevar, Schlömilch, Prym, and Vallier.

4. The logarithm integral, discussed by Euler, Soldner, Gauss, Bessel, Laguerre, and Stieltjes, which is applied in the expression of the number of primes less than a given number.

5. The cosine integral, studied by Schlömilch and Besso.

The first result obtained in the present paper is that *all the functions above named can be derived by specialization and transformation from a single new function.*

This function is denoted by $W_{k,m}(z)$, and the various functions above mentioned arise from it by taking special relations between the parameters k and m, and transforming the functions thus obtained: thus, the functions of the parabolic cylinders are derived by taking $m = \frac{1}{4}$, the incomplete gamma function is derived by taking $k + m = -\frac{1}{2}$, and so on. It is further shown that the Bessel functions are also cases of the same function $W_{k,m}(z)$.

It is evident, therefore, that if a general theory can be found for the function $W_{k,m}(z)$, then all the properties of the special functions in question can be deduced as mere corollaries. This is effected in § 3 of the present paper, where it is shown that *the function $W_{k,m}(z)$ is really a limiting case of the hypergeometric function* (some of the exponents of the hypergeometric function becoming infinite in a certain way), and that it possesses a theory which can be derived from that of the hypergeometric function. It appears, however, that the function $W_{k,m}(z)$ has, in the course of the limit process by which it is obtained, acquired certain new properties which are not possessed by the parent hypergeometric function; these properties are given.

The general family of functions $W_{k,m}(z)$ includes other classes of functions besides the five types of known functions already mentioned, and attention is drawn to some of these classes which have not hitherto been investigated.

§ 2. *Expression of the Parabolic Cylinder Functions, Error Function, Incomplete Gamma Functions, Logarithm Integral, and Cosine Integral by means of a Single Function.*

We now proceed to define a function in terms of which the various known functions already mentioned can be expressed. The function in question will be denoted by $W_{k,m}(z)$, and will be defined for all values of z (real or complex) by the equation

$$W_{k,m}(z) =$$

$$\frac{\Gamma(k+\frac{1}{2}-m)}{2\pi} e^{-\frac{1}{2}z+\frac{1}{2}i\pi} z^k \int (-t)^{-k-\frac{1}{2}+m} \left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}dt,$$

where the path of integration begins at $t = +\infty$, and after encircling the point $t = 0$ in the counter-clockwise direction returns to $t = +\infty$ again. The t-plane is supposed to be dissected by a cut from $t = 0$ to $t = +\infty$, and the expression $(-t)^{-k-\frac{1}{2}+m}$ is defined to mean

$$e^{(-k-\frac{1}{2}+m)\log(-t)},$$

where the real value of $\log(-t)$ is taken when t is real and negative.

When the real part of $-k-\frac{1}{2}+m$ is positive, the path of integration can be deformed so as to coincide with the real axis in the t-plane, and then (as is easily seen) the complex integral can be replaced by the real integral

$$W_{k,m}(z) =$$

$$\frac{1}{\Gamma(-k+\frac{1}{2}+m)} z^k e^{-\frac{1}{2}z} \int_0^\infty t^{-k-\frac{1}{2}+m} \left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t} dt.$$

We shall now show that the functions of the parabolic cylinder, error function, etc., can be expressed in terms of this function $W_{k,m}(z)$.

1. *The functions of the parabolic cylinder.*

The standard function associated with the parabolic cylinder in harmonic analysis is *

$$D_n(z) = \frac{2^{\frac{1}{2}n-2}\Gamma\left(\frac{n+1}{2}\right)}{\pi i^n} z \int e^{-\frac{1}{4}z^2 t}(t-1)^{-\frac{1}{2}n-\frac{1}{2}}(t+1)^{\frac{1}{2}n} dt,$$

where the integration is taken along a path which begins and ends at infinity in the direction which makes $z^2 t$ positive, and which encircles the point $t = 1$. This can be written

$$z^{\frac{1}{2}} D_{2k-\frac{1}{2}}(\sqrt{2}z) = \frac{2^{k-\frac{7}{4}}\Gamma(k+\frac{1}{4})}{\pi i^{2k-\frac{1}{2}}} z^{\frac{3}{4}} \int e^{-\frac{1}{2}zt}(t-1)^{-k-\frac{1}{4}}(t+1)^{k-\frac{1}{4}} dt,$$

and on writing $t = 1 + 2u/z$ in the integral and comparing with the definition of the function $W_{k,m}(z)$, we have

$$z^{\frac{1}{2}} D_{2k-\frac{1}{2}}(\sqrt{2}z) = 2^{k-\frac{1}{4}} W_{k,\frac{1}{4}}(z).$$

It follows that *the parabolic cylinder functions are expressed in terms of the function* $W_{k,m}(z)$ *by this equation,* and in fact correspond to the particular case $m = \frac{1}{4}$.

2. *The error function.*

The error function is defined by the integral

$$\int_z^\infty e^{-t^2} dt.$$

This function can be expressed in terms of the function $W_{k,m}(z)$. For from the definition of $W_{k,m}(z)$ we have

$$W_{-\frac{1}{4},\frac{1}{4}}(z^2) = z^{\frac{1}{2}} e^{-\frac{1}{4}z^2} \int_0^\infty \left(1+\frac{t}{z^2}\right)^{-\frac{1}{2}} e^{-t} dt$$

$$= 2z^{\frac{3}{2}} e^{-\frac{1}{2}z^2} \int_1^\infty e^{z^2(1-v^2)} dv, \qquad \text{where } v^2 = 1 + \frac{t}{z^2},$$

* Cf. a paper by the author in *Proc. Lond. Math. Soc.*, vol. 35, p. 417.

$$= 2z^{\frac{1}{4}}e^{\frac{1}{2}z^2}\int_z^\infty e^{-s^2}ds, \qquad \text{where } s = vz.$$

It follows that *the error function*

$$\int_z^\infty e^{-t^2}\,dt$$

can be expressed in terms of the function $W_{k,m}(z)$ *in the form*

$$\tfrac{1}{2}z^{-\frac{1}{4}}e^{-\frac{1}{2}z^2}W_{-\frac{1}{4},\frac{1}{4}}(z^2).$$

Certain other integrals discussed by Riemann in connection with the conduction of heat, *e. g.*, the integral

$$\int_a^b e^{-t^2-z^2/t^2}dt$$

can be evaluated in terms of the error function, and so in terms of the function $W_{k,m}(z)$.

3. *The incomplete gamma function.*

The incomplete gamma function is denoted by $\gamma(n, z)$, and is defined by the equation

$$\gamma(n, z) = \int_0^z t^{n-1}e^{-t}dt.$$

In order to express this in terms of the function $W_{k,m}(z)$, we observe that from the definition of the latter function we have

$$W_{\frac{1}{2}(n-1),\frac{1}{2}n}(z) = z^{\frac{1}{2}(n-1)}e^{-\frac{1}{2}z}\int_0^\infty \left(1+\frac{t}{z}\right)^{n-1}e^{-t}dt$$

$$= z^{\frac{1}{2}(1-n)}e^{\frac{1}{2}z}\int_z^\infty s^{n-1}e^{-s}ds, \qquad \text{where } s = t + z,$$

$$= z^{\frac{1}{2}(1-n)}e^{\frac{1}{2}z}\{\Gamma(n) - \gamma(n, z)\}.$$

It follows that *the incomplete gamma function can be expressed in terms of the function* $W_{k,m}(z)$ *by the relation*

$$\gamma(n, z) = \Gamma(n) - z^{\frac{1}{2}(n-1)}e^{-\frac{1}{2}z}W_{\frac{1}{2}(n-1),\frac{1}{2}n}(z).$$

4. *The logarithm integral.*

The logarithm integral $li(z)$ is defined by the equation

$$li(z) = \int_0^z \frac{dt}{\log t}.$$

The name was given by Soldner, and tables of the function were given by Bessel and Gauss.

Writing $z = e^{-\xi}$ and $t = e^{-u}$, we have

$$li(e^{-\xi}) = \int_{\infty}^{\xi} \frac{e^{-u}du}{u}$$

$$= -\frac{e^{-\xi}}{\xi}\int_0^{\infty}\left(1+\frac{s}{\xi}\right)^{-1} e^{-s}ds, \text{ where } s = u - \xi.$$

But

$$W_{-\frac{1}{2},0}(\xi) = \xi^{-\frac{1}{2}}e^{-\frac{1}{2}\xi}\int_0^{\infty}\left(1+\frac{t}{\xi}\right)^{-1} e^{-t}dt.$$

Therefore

$$li(e^{-\xi}) = -\xi^{-\frac{1}{2}}e^{-\frac{1}{2}\xi}W_{-\frac{1}{2},0}(\xi),$$

or *the logarithm integral is expressed in terms of the function* $W_{k,m}(z)$ *by the relation*

$$li(z) = -(-\log z)^{-\frac{1}{2}}z^{\frac{1}{2}}W_{-\frac{1}{2},0}(-\log z).$$

5. *The cosine integral.*

The cosine integral $Ci(z)$ is defined by the equation

$$Ci(z) = \int_{\infty}^{z}\frac{\cos t}{t}dt.$$

It can be expressed in terms of the function $W_{k,m}(z)$ in the following way. We have

$$Ci(z) = \tfrac{1}{2}\int_{\infty}^{z}\frac{e^{it}dt}{t} + \tfrac{1}{2}\int_{\infty}^{z}\frac{e^{-it}dt}{t}.$$

Putting $t = is$ in the first of these integrals, and $t = -is$ in the second, and remembering that

$$\int_{\pm i\infty}^{\infty}\frac{e^{-s}ds}{s}$$

is zero when the path of integration is wholly at infinity, we have

$$Ci(z) = \tfrac{1}{2}\int_{\infty}^{-iz}\frac{e^{-s}ds}{s} + \tfrac{1}{2}\int_{\infty}^{-iz}\frac{e^{-s}ds}{s}$$

$$= \tfrac{1}{2}li(e^{iz}) + \tfrac{1}{2}li(e^{-iz})$$

$$= \tfrac{1}{2}z^{-\frac{1}{2}}e^{\frac{1}{2}iz+i\pi/4}W_{-\frac{1}{2},0}(-iz) + \tfrac{1}{2}z^{-\frac{1}{2}}e^{-\frac{1}{2}iz-i\pi/4}W_{-\frac{1}{2},0}(iz),$$

which expresses the cosine integral in terms of the function $W_{k,m}(z)$.

6. It may be added that the Bessel functions can without much difficulty be shown to be likewise expressible in terms of the function $W_{k,m}(z)$. The actual relation will be found to be

$$J_m(z) = (2\pi z)^{-\frac{1}{2}}\{e^{(m+\frac{1}{2})i\pi/2}W_{0,m}(2iz) + e^{-(m+\frac{1}{2})i\pi/2}W_{0,m}(-2iz)\}.$$

§ 3. *Properties of the Function* $W_{k,m}(z)$.

We shall now establish the principal properties of the function $W_{k,m}(z)$.

1. *The differential equation satisfied by the function* $W_{k,m}(z)$. If we write

$$v = \int t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m}e^{-t}dt,$$

where the integration is taken along a path which begins at $t=\infty$, encircles the point $t=0$, and returns to $t=\infty$ again, we have

$$\begin{aligned}\frac{d^2v}{dz^2} + \left(\frac{2k}{z}-1\right)\frac{dv}{dz} + \frac{\frac{1}{4}-m^2+k(k-1)}{z^2}v \\ = \frac{k-\frac{1}{2}+m}{z^2}\int t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{k-3/2+m}e^{-t}\left\{t\left(1+\frac{t}{z}\right)\right. \\ \left. + k-\tfrac{1}{2}-m+(1-2m)\frac{t}{z}\right\}dt \\ = -\frac{k-\frac{1}{2}+m}{z^2}\int d\left\{t^{-k+\frac{1}{2}+m}\left(1+\frac{t}{z}\right)^{m-3/2+k}e^{-t}\right\};\end{aligned}$$

and this is zero, since the integrand is the perfect differential of a function which is zero at the terminal points of the path of integration. It follows that the function v satisfies the differential equation

$$\frac{d^2v}{dz^2} + \left(\frac{2k}{z}-1\right)\frac{dv}{dz} + \frac{\frac{1}{4}-m^2+k(k-1)}{z^2}v = 0.$$

Now from the definitions of $W_{k,m}(z)$ and of v, we have

$$W_{k,m}(z) = \text{constant} \times e^{-\frac{1}{2}z}z^k v,$$

and on substituting this value for v in the differential equation we see that $W_{k,m}(z)$ *satisfies the equation*

$$\frac{d^2W}{dz^2} + \left(-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right) W = 0.$$

It may be noted that any differential equation of the form

$$\frac{d^2y}{dz^2} + \left(a + \frac{b}{z} + \frac{c}{z^2}\right) y = 0$$

can be brought to this form, and therefore solved in terms of the functions $W_{k,m}(z)$.

2. *Reduction of the function* $W_{k,m}(z)$ *to a limiting case of the hypergeometric function.*

The functions $W_{k,m}(z)$ have in general an irregular singularity (using the word "irregular" in the sense customary in the theory of linear differential equations) at $z = \infty$, and are therefore not members of the family of hypergeometric functions, which are characterized by the fact that their singularities are all regular. In spite of this, however, the functions $W_{k,m}(z)$ can be derived from the hypergeometric function by a limit process, namely, making the exponents at two of its singularities infinite, and making the two singularities in question to coalesce at infinity.

For the differential equation corresponding to the hypergeometric function

$$P\begin{Bmatrix} 0 & \infty & c & \\ \frac{1}{2}+m & -c & c-k & z \\ \frac{1}{2}-m & 0 & k & \end{Bmatrix}$$

is easily found to tend, for $c = \infty$, to the limiting form

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} + \left[\frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right] y = 0,$$

and on putting $y = e^{-\frac{1}{2}z}v$ in this, we obtain

$$\frac{d^2v}{dz^2} + \left[-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right] v = 0,$$

which is the differential equation satisfied by the function $W_{k,m}(z)$.

3. *Other solutions of the differential equation of the function* $W_{k,m}(z)$.

It is clear that the differential equation of the function $W_{k,m}(z)$ is unchanged if m be replaced by $-m$; it is also unchanged if k be replaced by $-k$, provided z be replaced by $-z$ at the same time. Hence *the four functions*

$$W_{k,m}(z), \quad W_{k,-m}(z), \quad W_{-k,m}(-z), \quad W_{-k,-m}(-z),$$

are solutions of the differential equation

$$\frac{d^2y}{dz^2} + \left(-\tfrac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) y = 0.$$

Cases of this result are the well-known theorems that $J_n(z)$ and $J_{-n}(z)$ each satisfy Bessel's equation, and that $D_n(z)$ and $D_{-n-1}(iz)$ each satisfy the equation of the parabolic cylinder.

4. *The asymptotic expansion of* $W_{k,m}(z)$.

We have

$$W_{k,m}(z) =$$

$$\frac{\Gamma(k + \frac{1}{2} - m)}{2\pi} e^{-\frac{1}{2}z + \frac{1}{2}k\pi i} z^k \int (-t)^{-k-\frac{1}{2}+m} \left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t} dt,$$

and it is easily seen that the divergent series which is obtained by expanding the quantity $(1 + t/z)^{k-\frac{1}{2}+m}$ by the binomial theorem, and then integrating term by term, is the asymptotic expansion of the integral for large positive values of z. This series is

$$e^{-\frac{1}{2}z} z^k \left\{1 + \frac{m^2 - (k - \frac{1}{2})^2}{1!\, z} + \frac{\{m^2 - (k - \frac{1}{2})^2\}\{m^2 - (k - \frac{3}{2})^2\}}{2!\, z^2} + \cdots\right\}$$

This is therefore the asymptotic expansion of the function $W_{k,m}(z)$.

The well-known asymptotic expansions of the logarithm integral, error function and Bessel functions, and the asymptotic expansion of the parabolic cylinder functions given by the author in *Proceedings of the London Mathematical Society*, volume 35, page 417, are cases of this expansion.

5. *Further properties of the functions* $W_{k,m}(z)$.

The remaining principal properties of functions of the type $W_{k,m}(z)$ will be stated without proof, as the reader will without difficulty be able to supply demonstrations.

(*a*) *A second definite integral expression for* $W_{k,m}(z)$ is

$$W_{k,m}(z) =$$

$$\tfrac{1}{2}\pi\Gamma(k+\tfrac{1}{2}+m)e^{-\frac{1}{2}z+i\pi/2}z^{k}\int(-t)^{-k-\frac{1}{2}-m}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}-m}e^{-t}dt,$$

the integration being taken as before along a path which begins and ends at $t=+\infty$ and encircles $t=0$.

(*b*) The function $W_{k,m}(z)$ *degenerates into an elementary function* when $k-\frac{1}{2}\pm m$ is a positive integer or zero; the theorem that the Bessel function degenerates into an elementary function when its order is half an odd integer is evidently a particular case of this result.

(*c*) The function $W_{k,m}(z)$ *degenerates into the indefinite integral of an elementary function* when $k-\frac{1}{2}\pm m$ is a negative integer; the incomplete gamma function, logarithm integral, error function, and cosine integral, are all instances of this theorem.

(*d*) Considering the case in which $W_{k,m}(z)$ degenerates into an elementary function, if $W_{k_1,m}(z)$ and $W_{k_2,m}(z)$ are two functions of this kind, with different parameters k but the same parameter m, *the integral property*

$$\int_0^\infty W_{k_1,m}(z)\,W_{k_2,m}(z)\frac{dz}{z}=0$$

can be established.

(*e*) *Recurrence formulas* also exist: if there are any three functions $W_{k,m}(z)$ such that the corresponding values of k differ by integers, and the values of m also differ by integers, then the three functions are connected by a linear relation, the coefficients in which are polynomials in z.

6. *New classes of functions.*

As has been seen above, a number of functions which are really members of the family of functions $W_{k,m}(z)$ have in the past been separately discovered and investigated. There are other members of this family which have not hitherto been noticed, but which give promise of interesting properties. Among these may be mentioned the families of functions for which $m=0$,

and those for which $m = \frac{1}{2}$. With regard tc the former of these classes, for example, it may be shown that if an arbitrary function $f(z)$ be expanded in the form

$$f(z) = a_0 W_{1/2,\,0}(z) + a_1 W_{3/2,\,0}(z) + a_2 W_{5/2,\,0}(z) + \cdots,$$

then the coefficients are given by the relation

$$a_n = \frac{1}{(n!)^2} \int_0^\infty f(z)\, W_{n+\frac{1}{2}}(z)\, \frac{dz}{z}.$$

TRINITY COLLEGE, CAMBRIDGE,
July, 1903.

ON THE FACTORING OF LARGE NUMBERS.

BY PROFESSOR F. N. COLE.

(Read before the American Mathematical Society, October 31, 1903.)

1. IN resolving a large number N into its prime factors, a table of quadratic remainders of N can be made to render efficient service in several different ways. For a twenty-two place N, the remainders of the table may be restricted to products of about seventy of the smallest prime numbers available. If, for example, N is always expressed in the form $N = x^2 \pm a$, with variable x and a, the remainders $\mp a$ will contain only those primes of which N is quadratic remainder. By a gradual elimination of common factors from the remainders $\mp a$, we finally obtain a table of remainders not admitting further reduction. Other forms for expressing N, such as $N = 2x^2 \pm b$, are of course often advantageous, according to circumstances. If the final table of remainders consists entirely of the individual primes employed, each with the proper sign + or −, N is undoubtedly a prime number*; in fact a much smaller sequence of prime remainders would suffice to justify this conclusion, the wide range specified above being required only to ensure the success of the preliminary elimination process. If, on the other hand, it is found impossible, on repeated attempt,

* For an interesting discussion of the use of the table of remainders see P. Seelhoff, *Zeitschrift für Mathematik und Physik*, vol. 31 (1886), pp. 166, 174, 306.

to isolate the prime numbers as remainders, there is a strong presumption amounting almost to certainty that N is compound.

By the aid of the quadratic remainders of the table it is now theoretically possible to construct the factors of N synthetically. Occasionally the form of these factors is so far known in advance that it is not an over serious task to sift all possible cases up to several millions. In the sifting process each known quadratic remainder reduces the number of possible factors of N by one-half. Thus, with twenty-four remainders at hand, sixteen million numbers would be reduced to one or two. In the practice a few of the known remainders would be employed to establish linear series $lx + m$ with variable x; the other remainders would then be used to sift the values of x.

If, however, the smallest factor of N has nine or more places, a much more expeditious method is to employ the known quadratic remainders to reduce N to the form

$$x^2 - y^2 = [\tfrac{1}{2}(u + v)]^2 - [\tfrac{1}{2}(u - v)]^2 = uv,$$

the factors u and v being so chosen that their difference $u - v$ is as small as possible. The advantage of this method is the following: If the table of quadratic remainders is constructed solely from resolutions $N = x^2 + a$, u and v are quadratic remainders of every prime remainder r of the table, and as the product $uv = N$ is known, the symmetric function $u + v$ admits at most $\frac{1}{4}(r + 3)$ values, mod r. Every known prime remainder r therefore cuts down the number of possible values of $u + v$ to about *one-fourth* their number. Other limitations on the value of $u + v$ and $u - v$ may also occur to facilitate the process.

2. Next to the Fermat numbers $2^{2^n} + 1$, the numbers of the form $2^p - 1$, where p is a prime, have attracted the greatest attention as regards their resolution into prime factors. Some curious misinformation in regard to these numbers was published by Mersenne in the preface of his Cogitata physico-mathematica (1644).* A table of their prime factors, as far as $2^{59} - 1$, is most easily accessible in Lucas's article: "Théorie des fonctions numériques simplement périodiques," in volume 1 of the *American Journal of Mathematics*.† The smallest factors are given, so far as known, up to $p = 257$, by Lucas on page 375 of his Théorie des nombres, and on the following page

* *Cf.* Lucas, Théorie des nombres, p. 376.

† The table occurs on pp. 239–240 of the volume.

of the same work the author announces that his calculations indicate that $2^{67}-1$ and $2^{89}-1$ are compound. It is known that $2^{61}-1$ is a prime number.* In the article cited above, Lucas presented a brilliant method for ascertaining whether a given number is prime. His computations just mentioned are doubtless based on this method, which however does not furnish the means of actually finding the factors of the number examined.

3. For the number

$$2^{67}-1 = 147573952589676412927$$

I found the following apparently irreducible quadratic remainders:

2, -3, -7, 13, -23.53, 37, 41, 61, -67, -71, 23.83, 89, 97, 101, -23.113, -127, 137, 23.151, -23.157, 23.167, 173, 181, 23.191, -23.193, ...

The conclusion was inevitable, especially in view of Lucas's prediction, that $2^{67}-1$ possessed precisely two prime factors greater than 1.

It is well known that every factor of 2^p-1, p being a prime, is of the forms $kp+1$ and $8l\pm 1$. Every factor of $2^{67}-1$ is therefore of the forms $536k+1$, $536l+135$, which are further specialized to $1608k+1$ and $1608l+1207$ by virtue of their common quadratic remainder -3. With the help of the other small prime remainders, fortified by the presumption that -23 and therefore 53, -83, ... were non-remainders, I had no difficulty in sifting the first sixteen million natural numbers, but the search proved without result.

Turning then to the resolution

$$2^{67}-1 = [\tfrac{1}{2}(u+v)]^2 - [\tfrac{1}{2}(u-v)]^2,$$

and noting that $u-v$ is divisible by 3 and 67, I found the following congruences for determining $\frac{1}{2}(u+v)$:

$$\begin{aligned}\tfrac{1}{2}(u+v) &\equiv 671 \pmod{67^2} \equiv 0 \pmod 8 \equiv 1, 4^+ \pmod 5\\ &\equiv 1, 3^+ \pmod 7 \equiv 0, 1^+, 12 \pmod{13}\\ &\equiv 10, 37, 46^+, 64 \pmod{81}.\end{aligned}$$

* P. Seelhoff, l. c., p. 174. Seelhoff's papers cited above contain several errata, and his proof that $2^{61}-1$ is a prime number is hardly satisfactory. I have, however, verified the fact by the actual computation of a sequence of prime remainders. The verification has also been made by others; see Weber-Wellstein: Encyclopaedie der Elementar-Mathematik, p. 48.

The cases marked $^+$ give

$$\tfrac{1}{2}(u+v) = 1323536760x + 1160932384.$$

Sifting with 23, 37, 41, 53, 61, I find $x = 287$ remaining. A few further tests suffice to show that the goal is near. And, in fact,

$$287.1323536760 + 1160932384 = 381015982504,$$

and

$$2^{67} - 1 = 381015982504^2 - 380822274783^2$$
$$= 193707721 \times 761838257287.$$

COLUMBIA UNIVERSITY,
October, 1903.

NOTE ON THE p-DISCRIMINANT OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, August 31, 1903.)

IN a fundamental memoir * Darboux has proved that the resultant $g(x, y) = 0$ obtained by the elimination of $p = dy/dx$ between

$$\phi(x, y, p) = 0 \ (1), \quad \text{and} \quad \partial\phi/\partial p \ (2),$$

where ϕ is a polynomial in x, y, p, represents in general the locus of the cusps of the integral curves.†

For this purpose Darboux first proves that the resultant obtained by the elimination of p between

$$\phi(x, y, p) = 0 \ (3), \quad \text{and} \quad \frac{\partial\phi}{\partial x} + p\,\frac{\partial\phi}{\partial y} = 0 \ (4)$$

represents in general the locus of the points of inflexion of the integral curves.

By purely geometric reasoning, applying the principle of duality, Darboux then concludes that to this theorem corresponds dualistically the theorem in connection with equations (1) and (2) as stated above.

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* "Sur les solutions singulières des équations aux derivées ordinaires du premier ordre." *Bull. des Sciences Math. et Astron.*, vol. 4, pp. 158–176 (1873).

† For an analytic proof see Picard's Traité d'Analyse, vol. 3, pp. 529–534.

In this note I shall prove directly that to the condition (4) corresponds dualistically the condition (2).

Assume the circle

$$x^2 + y^2 = 1,$$

and the transformation by reciprocal polars with respect to this circle. Designating the coördinates of an arbitrary point of the plane of the circle by (x, y) and of its transformed by (x_1, y_1) the relations hold *

$$x = \frac{-p_1}{y_1 - x_1 p_1}, \quad y = \frac{1}{y_1 - x_1 p_1}, \quad p = -\frac{x_1}{y_1} \tag{5}$$

$$(p = dy/dx, \quad p_1 = dy_1/dx_1).$$

By this transformation $\phi(x, y, p) = 0$ is transformed into another linear differential equation of the same order

$$\phi_1(x_1, y_1, p_1) = 0,$$

and

$$\frac{\partial \phi}{\partial x} + p \frac{\partial \phi}{\partial y} = 0$$

into an expression resulting next from

$$\frac{\partial \phi}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial x} + p \frac{\partial \phi}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial y} = 0,$$

or

$$\frac{\partial \phi}{\partial \phi_1} \left\{ \frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial \phi_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial x} + \frac{\partial \phi_1}{\partial p_1} \cdot \frac{\partial p_1}{\partial x} + p \left(\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial y} + \frac{\partial \phi_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial y} + \frac{\partial \phi_1}{\partial p_1} \cdot \frac{\partial p_1}{\partial y} \right) \right\} = 0. \tag{6}$$

But according to (5)

$$\frac{\partial x_1}{\partial x} = -\frac{p^2}{(y - xp)^2}, \quad \frac{\partial y_1}{\partial x} = \frac{p}{(y - xp)^2}, \quad \frac{\partial p_1}{\partial x} = -\frac{1}{y},$$

$$\frac{\partial x_1}{\partial y} = \frac{p}{(y - xp)^2}, \quad \frac{\partial y_1}{\partial y} = -\frac{1}{(y - xp)^2}, \quad \frac{\partial p_1}{\partial y} = \frac{x}{y^2}.$$

Substituting these values in (6) and reducing, we get from (6)

$$\frac{\partial \phi}{\partial \phi_1} \cdot \frac{\partial \phi_1}{\partial p_1} \left(-\frac{1}{y} + p \frac{x}{y^2} \right) = 0. \tag{7}$$

* S. Lie, Berührungstransformationen, vol. 1, p. 23.

But generally $\partial\phi/\partial\phi_1$ and $-1/y + px/y^2$ do not vanish, so that (7) reduces to the condition $\partial\phi_1/\partial p_1 = 0$. Hence the theorem:

To the simultaneous equations

$$\phi(x, y, p) = 0, \quad \frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0$$

correspond dualistically, the equations

$$\phi_1(x_1, y_1, p_1) = 0, \quad \partial\phi_1/\partial p_1 = 0.$$

The same result would be obtained by operating with the analytic expressions for the most general dualistic transformation. The special transformation (5) was chosen for the sake of simplicity.

UNIVERSITY OF COLORADO,
August, 1903.

HYDRODYNAMIC ACTION AT A DISTANCE.

Vorlesungen über hydrodynamische Fernkräfte nach C. A. BJERKNES'S *Theorie.* Von V. BJERKNES. 2 Bände, 8vo. Leipzig, J. A. Barth, 1900–1902. Bd. I, xvi + 338 pp., 40 figs. Bd. II, xvi + 316 pp., 60 figs.

C. A. BJERKNES is dead. The news is scarcely yet spread over the scientific world. No more fitting time could be found for calling attention to his life-work on hydrodynamic action at a distance. Pupil of Dirichlet at Göttingen, professor of mathematics and physics at the university in Christiania, ardent admirer and follower of Faraday and Maxwell, Bjerknes more than twenty years ago had developed practically to completion a theory which never has received much attention owing to the manner of its publication. That the work is now before the public in a complete, accessible, and easily intelligible form is due to the editorial patience of the son, V. Bjerknes.

The first volume, designed primarily for mathematicians, contains the theoretical development of the mutual actions of pulsating and oscillating spheres immersed in a common incompressible perfect fluid. Fortunately the mathematical analysis is so elementary and so carefully explained that the most meager training amply suffices for its comprehension. In

the second volume, which will appeal almost exclusively to physicists, are found the same laws and the formulas governing them, deduced experimentally much in the manner in which the laws of electricity and electromagnetism were discovered and formulated by Coulomb and Faraday. The remarkable fact is that the experimental results are numerically so large. The experiments, which were hastily shown to some of us at the Abel centenary, exhibited the same order of magnitude, the same degree of conclusiveness as the corresponding experiments in the fields of electricity and magnetism. It is fortunate that the firm of Ferdinand Ernecker has undertaken to prepare for the market apparatus similar to that constructed by Bjerknes for use in his private laboratory. A description of the apparatus is found in volume II of the work under review.

To the untrained intuition the phenomena connected with hydrodynamic actions can yield nothing but puzzles. That bodies moving immersed in a common fluid should produce some actions and reactions on one another is readily granted; what sort of actions these may be is not easily divined. For example it is well known that a sphere, free to move and uninfluenced by external forces, may remain at rest in a constant current. That this apparently contradicts all experience and all common sense is due to the imperfection of the spheres and fluids with which we ordinarily come in contact. Under certain circumstances it may even come about that a sphere moves up stream against a strong current. This hydrodynamic paradox, as it is called, has been exhibited far and wide in toys placed upon the market to the mystification of more than children. Even the trained physicist finds it safe to trust rather to analysis than to intuition.

The analysis in the case of arbitrary bodies is far too difficult to follow out. In the case of anchor rings, to which the cosmographic speculations of Lord Kelvin have given such interest, the theory of mutual actions and reactions has not yet reached a satisfactory development, notwithstanding the classic memoir published more than twenty years ago by J. J. Thomson. The present authors when dealing with hydrodynamic actions at a distance restrict themselves for the sake of simplicity to those actions arising from or experienced by spheres pulsating or oscillating with a definite period and separated from one another by a distance great in comparison with the radius of the spheres. If the distance between the spheres be

taken as unity, the radius will be a small number, the powers of which may serve as a measure for the different orders of approximation and the different orders of magnitude of the actions which take place. The analysis is carried out in such detail that actions of magnitude down to and including the seventh order are taken into account.

If a sphere

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2$$

pulsate about its fixed center (a, b, c), the radius d and the volume $E=\frac{4}{3}\pi d^3$ are functions of the time t. If the pulsation take place periodically with period τ, the quantity

$$(\dot{E})_m=\pm\sqrt{\frac{1}{\tau}\int_t^{t+\tau}(\dot{E})^2dt}, \quad \text{where} \quad \dot{E}=\frac{dE}{dt},$$

is the quadratic mean of the rate of change of volume and may be regarded as defining the *intensity of pulsation*. The sign + or − is chosen according as the sphere be dilating or contracting at the origin of time.

During the pulsations the particles of the fluid in which the sphere is immersed move radially to and fro. The magnitude and the direction of the velocity of this displacement at any given point of the fluid are derivable from a *velocity potential*

$$\phi=-\frac{\dot{E}}{4\pi r}.$$

It is found convenient to express the rate of cubic dilation $\dot{E}$ as the product of the mean rate $(\dot{E})_m$ and a periodic function $\dot{f}(t)$ which is written as a time derivative for reasons of analogy.

$$\dot{E}=(\dot{E})_m\dot{f}(t).$$

Then

$$\phi=-\dot{f}(t)\frac{(\dot{E})_m}{4\pi r}.$$

Thus the velocity potential is divided into two factors, of which the first, depending on the time, is purely kinematic, while the second is purely geometric.

To a first approximation the velocity potential of a number of pulsating spheres is the sum of the potentials due to the individual spheres

$$\phi=-\sum_i\dot{f}_i(t)\frac{(\dot{E}_i)_m}{4\pi r_i}.$$

The interesting case to consider is that of synchronism in which the periodic functions $\dot{f}_i(t)$ appertaining to the different spheres are identical. The potential again becomes separable into a kinematic and a geometric factor

$$\phi = -\dot{f}(t) \sum_i \frac{(\dot{E}_i)_m}{4\pi r_i}.$$

By comparison with the potential function Φ due to a system of magnetic poles magnetized with intensities I_i,

$$\Phi = +\sum_i \frac{I_i}{4\pi r_i},$$

it appears that the lines of flow set up in a liquid by a system of spheres pulsating with intensities $(\dot{E}_i)_m$ are identical with the lines of force set up by a system of magnetic poles of intensities $I_i = -(\dot{E}_i)_m$ distributed geometrically in the same manner as the spheres.

Further magnetic analogies are immediately derivable by considering a system of two spheres pulsating with equal intensities and opposite phases. The potential is

$$\phi = -\dot{E}\left(\frac{1}{4\pi r_1} - \frac{1}{4\pi r_2}\right).$$

In case the vector $\mathbf{r}_{12}$ joining the center of the first or "positive" sphere to that of the second or "negative" sphere be infinitesimal and the radii of the spheres be infinitesimals of higher order, the system is called a *doublet*. The potential of a doublet takes the form

$$\phi = -\frac{\dot{\mathbf{S}} \cdot \mathbf{r}}{4\pi r^3},^*$$

where $\dot{\mathbf{S}} = \dot{E}\mathbf{r}_{12}$ is by definition the vector *kinematic moment of action* of the doublet, and $\mathbf{r}$ is the radius vector from the center of the doublet to any point in space. The corresponding potential for a "small" magnet is

$$\Phi = +\frac{\mathbf{I} \cdot \mathbf{r}}{4\pi r^3},$$

where $\mathbf{I}$ is the vector intensity of magnetization.

*The notation for vectors which we adopt here is that of Gibbs. See Vector Analysis by E. B. Wilson, reviewed in the BULLETIN, February, 1902, by A. Ziwet.

Passing to the case of a rigid sphere moving with a velocity $\dot{s}$ of translation, the potential is found to be

$$\phi = -\tfrac{1}{2}\frac{d^3}{r^3}\dot{s}\cdot\mathbf{r}.$$

The vector $\dot{S} = 2\pi d^3\dot{s} = \tfrac{3}{2}E\dot{s}$ may be defined as the kinematic moment of action and the potential again takes the form

$$\phi = -\frac{\dot{S}\cdot\mathbf{r}}{4\pi r^3}.$$

With these conventions appears the remarkable and fundamental fact that the velocity fields set up by a doublet of pulsating spheres and by a sphere moving with translation are identical, provided only that the kinematic moment of action is the same in both cases. In the case of translation the moment of action is three halves the product of the volume by the velocity; for the doublet it is the product of the intensity of pulsation by the distance between the spheres. To treat an oscillating sphere it is necessary to replace the velocity of translation and the moment of action by their mean values.

The parameters a, b, c, d which define the position of a sphere suffer changes of which the velocities are $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$. These parameters and their velocities are divisible into two parts, one slow and progressive, the other fast and oscillating about some mean value. Let

$$a = a^0 + a^1, \quad b = b^0 + b^1, \quad c = c^0 + c^1, \quad d = d^0 + a^1,$$

$$\dot{a} = \dot{a}^0 + \dot{a}^1, \quad \dot{b} = \dot{b}^0 + \dot{b}^1, \quad \dot{c} = \dot{c}^0 + \dot{c}^1, \quad \dot{d} = \dot{d}^0 + \dot{d}^1.$$

As the spheres are not subject to permanent large changes in magnitude, $\dot{d}^0$ is zero and d^0 is constant. a^0, b^0, c^0 change slowly and the velocities $\dot{a}^0$, $\dot{b}^0$, $\dot{c}^0$ are comparable to the velocities of finite masses of matter — the velocities encountered in molar physics. a^1, b^1, c^1, d^1 change rapidly, but remain always small in comparison with d^0, the radius of the sphere. The velocities $\dot{a}^1$, $\dot{b}^1$, $\dot{c}^1$, $\dot{d}^1$ may attain great values and are comparable to the velocities found in molecular physics. Evidently the linear means of the actual velocities a, b, c, d are identical with the linear means of the molar velocities $\dot{a}^0$, $\dot{b}^0$, $\dot{c}^0$, $\dot{d}^0$, and

the quadratic means are to a first approximation identical with the quadratic means of the molecular velocities $\dot{a}^1$, $\dot{b}^1$, $\dot{c}^1$, $\dot{d}^1$.

The potential of a system of spheres simultaneously undergoing pulsations and translations is

$$\phi = -\sum_i \frac{\dot{E}_i}{4\pi r_i} - \tfrac{1}{2}\sum_i \frac{\dot{d}_i^3}{r_i^3}\dot{\mathbf{s}}_i \cdot \mathbf{r}_i,$$

and may evidently be written in the form

$$\phi = \sum (A_i\dot{a}_i + B_i\dot{b}_i + C_i\dot{c}_i + D_i\dot{d}_i),$$

where A, B, C, D are functions of the geometric parameters a, b, c, d and are independent of the velocities $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$. In case of synchronous pulsations and oscillations the geometric functions A, B, C, D may be replaced by their linear mean values, that is, by these values when for a, b, c, d are substituted a^0, b^0, c^0, d^0, and the velocities $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$ may be replaced by their quadratic mean values $\dot{a}^1_m$, $\dot{b}^1_m$, $\dot{c}^1_m$, $\dot{d}^1_m$, which are equal to $\dot{a}^1_m$, $\dot{b}^1_m$, $\dot{c}^1_m$, $\dot{d}^1_m$. The potential then takes on a mean value

$$\phi_m = \sum_i (A_i^0\dot{a}^1_m + B_i^0\dot{b}^1_m + C_i^0\dot{c}^1_m + D_i^0\dot{d}^1_m).$$

The authors next pass to the discussion of an arbitrary velocity field, potential and solenoidal. The function ϕ is developed according to powers of $x-a$, $y-b$, $z-c$ into the form

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_n + \cdots.$$

Here

$$\phi_1 = \alpha(x-a) + \beta(y-b) + \gamma(z-c),$$

where α, β, γ are the values of the first partial derivatives of ϕ at the point a, b, c.

$$\phi_2 = \tfrac{1}{2}\dot{\alpha}_\alpha(x-a)^2 + \tfrac{1}{2}\dot{\beta}_\beta(y-b)^2 + \tfrac{1}{2}\dot{\gamma}_\gamma(z-c)^2 + \dot{\beta}_\gamma(y-b)(z-c) + \text{etc.},$$

where $\dot{\alpha}_\alpha$, $\dot{\beta}_\beta$, $\dot{\gamma}_\gamma$, $\dot{\beta}_\gamma$, etc., are values of the second partial derivatives of ϕ at the point a, b, c. In vector form the expansion may be written

$$\phi = \phi_0 + \mathbf{r}\cdot\nabla\phi + \mathbf{r}\cdot\nabla\nabla\phi\cdot\mathbf{r} + \cdots,$$

where

$$\nabla\cdot\nabla\phi = 0.$$

The term $\phi_1 = \mathbf{r} \cdot \nabla\phi$ defines a parallel field. The equipotential surfaces are planes; and the lines of flow, straight lines. The second term defines what Bjerknes calls a deformation field. The equipotential surfaces are quadrics to which the lines of flow are everywhere perpendicular.

After these preliminary considerations the authors take up the general kinematic actions and reactions between a sphere and a current. The potential due to the motion of a sphere immersed in a fluid originally at rest is divided into two parts; the internal, that of the motion of the sphere itself, and the external, that due to the motion induced in the fluid by the motion of the sphere. In the case of simple pulsations or oscillations these potentials are as given above. But as the field of the stream acts on the sphere, the relations become far more complicated and require a more elaborate treatment. If, in the general case, the interior potential of the sphere be

$$\Phi = -\tfrac{1}{2}\dot{d} + \Phi_1 + \cdots + \Phi_n + \cdots,$$

the exterior is

$$\phi = -\frac{d^2}{r}\dot{d} - \tfrac{1}{2}\frac{d^3}{r^3}\Phi_1 - \cdots - \frac{n}{n+1}\frac{d^{2n+1}}{r^{2n+1}}\Phi_n - \cdots$$

The potential of a stream in which a sphere at rest is immersed is likewise divided into two parts. If the original, that is to say incident, stream be defined by the potential

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_n + \cdots,$$

the total resulting stream has the potential

$$\phi = \phi_0 + \left(1 + \tfrac{1}{2}\frac{d^3}{r^3}\right)\phi_1 + \cdots + \left(1 + \frac{n}{n+1}\frac{d^{2n+1}}{r^{2n+1}}\right)\phi_n + \cdots,$$

of which

$$\tfrac{1}{2}\frac{d^3}{r^3}\phi_1 + \cdots + \frac{n}{n+1}\frac{d^{2n+1}}{r^{2n+1}}\phi_n + \cdots$$

is the *reaction* or *reflex* potential arising from the obstructing presence of the sphere at rest in the stream. It results that for a sphere moving in a current the total external potential takes the form

$$\phi = -\frac{d^3}{r}\dot{d} - \sum_{n=1}^{\infty}\frac{n}{n+1}(\Phi_n - \phi_n) + \sum_{n=1}^{\infty}\phi_n,$$

of which the first term arises directly from the pulsation, the second gives the difference between the potentials of the sphere and of the current, and the third is due to the incident stream. This formula serves as the starting point of a number of applications to the motion of spheres advancing and pulsating in parallel or deformation fields. The cuts which accompany the text are admirably drawn and instructive.

To pass to dynamic considerations, use is made of the formula for the pressure p at any point of the fluid, expressed in terms of the velocity potential ϕ, the pressure P which would exist if the system were at rest, and the density q,

$$p = P - q\frac{\partial \phi}{\partial t} - \tfrac{1}{2}q\nabla \cdot \nabla\phi.$$

By somewhat lengthy analysis, filled with approximations, the authors succeed in establishing the formula for the total (approximate) force acting upon the sphere,

$$\mathbf{F} = -q\frac{d}{dt}\{E(\tfrac{1}{2}\dot{\mathbf{s}} - \tfrac{3}{2}\dot{\sigma})\} - q\dot{E}\dot{\sigma} - \tfrac{3}{2}qE\{(\dot{\mathbf{s}} - \dot{\sigma}) \cdot \nabla\nabla\phi\}.$$

It is to be noted that σ is the velocity of the stream ϕ, and $\dot{\mathbf{s}}$ that of the sphere. This value for $\mathbf{F}$ is *exact* if the field ϕ be a parallel field, a deformation field, or a combination of both. In other cases the error depends on the degree of divergence of the field from the nearest possible combination of a parallel and a deformation field. In practice the divergence is so slight that the expression above differs from the exact value of $\mathbf{F}$ by a quantity of the eighth order at most and may consequently be regarded as quite satisfactory.

The equation of motion for a sphere of mass M immersed in the fluid and acted upon by an external force $\mathbf{f}$ is

$$M\mathbf{a} = M\ddot{\mathbf{s}} = \mathbf{F} + \mathbf{f}.$$

The integration of this equation gives the complete theory of hydrodynamic action at a distance. The special cases which arise, according as some of the terms of $\mathbf{F}$ are missing or as ϕ has a special form, yield interesting results easily confirmable by experience. These the authors consider first.

As the whole trend of the work is to exhibit the analogies to electricity and magnetism with especial regard to the point of view of Maxwell, there is great attention paid to the

medium, that is, the liquid in which the actions take place. The reciprocal of the density, or the specific volume as it is also called, carries the name of *mobility* and corresponds to the polarizability of the mediums in electricity and magnetism. The velocity is multiplied by the density q to give the *intensity of the field* in the fluid. The kinematic moment of action of a sphere is likewise multiplied by the density q to obtain the *dynamic moment of action*. The velocity potentials, ϕ in the fluid and Φ in the sphere, are replaced by the potentials of the field intensity, $q\phi$ and $Q\Phi$ respectively (where Q is the density of the sphere).

The total force $\mathbf{F}$ is separated into two parts. The force of *hydrodynamic induction*

$$\mathbf{F}_{\text{ind}} = -q\frac{d}{dt}\{E(\tfrac{1}{2}\dot{\mathbf{s}} - \tfrac{3}{2}\dot{\sigma})\}$$

is a perfect time derivative and derives its name from electric and magnetic analogies. The force of *hydrodynamic energy*

$$\mathbf{F}_{\text{en}} = -q\dot{E}\dot{\sigma} - \tfrac{3}{2}qE\{(\dot{\mathbf{s}} - \dot{\sigma}) \cdot \nabla\nabla\phi\}$$

differs in no essential way from the external force $\mathbf{f}$. The first integration of the equation of motion gives

$$\dot{\mathbf{s}} = -\frac{qE}{M}(\tfrac{1}{2}\mathbf{s} - \tfrac{3}{2}\sigma) + \frac{1}{M}\int_0^t (\mathbf{F}_{\text{en}} + \mathbf{f})\,dt.$$

The last term is the velocity generated by the energy forces external and internal, and will be denoted by $\dot{\mathbf{s}}_{\text{en}}$. Then

$$\dot{\mathbf{s}} = \frac{Q}{Q + \frac{1}{2}q}\dot{\mathbf{s}}_{\text{en}} + \frac{\frac{3}{2}q}{Q + \frac{1}{2}q}\dot{\sigma}.$$

As the greater number of the fields with which the authors deal are set up by pulsating and oscillating spheres, the value of $\dot{\sigma}$ oscillates about zero. This gives rise to a new nomenclature: The first term in $\dot{\mathbf{s}}$, above written, is called the *permanent* velocity; the second, the *temporary*. Subsequently the force of energy is likewise subdivided into the permanent force of energy $\mathbf{F}_{p\text{-en}}$, and the temporary

$$\mathbf{F}_{t\text{-en}} = -\tfrac{3}{2}q\frac{q - Q}{Q + \frac{1}{2}q}E\dot{\sigma} \cdot \nabla\nabla\phi.$$

It is evident that no temporary force of energy exists, unless the densities of the liquid and sphere differ. The force of induction is divided into the self-induction

$$\mathbf{F}_{s\text{-ind}} = -q\frac{d}{dt}\{E(\tfrac{1}{2}\dot{\mathfrak{s}})\},$$

which depends not at all on the stream but only on the sphere, and the foreign induction

$$\mathbf{F}_{f\text{-ind}} = +q\frac{d}{dt}\{E(\tfrac{3}{2}\dot{\sigma})\},$$

dependent upon the velocity of the stream and the volume of the sphere.

After a discussion, for about eighty pages, of the results obtainable from this subdivision of the total force into four partial forces, there follows a short chapter on rotary effects. As the fluid is frictionless, the individual spheres can not be set into rotation about their axes; but if a doublet or an oscillating sphere be considered, the line of centers of the spheres composing the doublet or the line of oscillation of the sphere is subject to change. The moment of rotation due to the force of energy is very simple,

$$-\tfrac{3}{2}qE\dot{\mathfrak{s}} \times \dot{\sigma},$$

and that due to the induction is

$$-q\frac{d}{dt}\{E(\mathbf{r}-\mathbf{r}_0) \times (\tfrac{1}{2}\dot{\mathfrak{s}} - \tfrac{3}{2}\dot{\sigma})\},$$

where $\mathbf{r}-\mathbf{r}_0$ is the vector drawn between the positions of the center of the sphere at two successive instants.

To this point the question has been to treat the mutual influence of a stream ϕ and a sphere. If now the stream be the result of the motions of other spheres the expressions above found for forces and moments will give the actions of the other (distant) spheres upon the sphere in question. As the stream ϕ must be a composition of a parallel and a deformation field if the formulas obtained may be considered as exact, and as the streams set up by other spheres do not in general come under this restricted category, the formulas for forces and moments are only approximate. The error, however, is very small — of the eighth order at most. The formulas are accurate down

to and including the seventh order. Thus if the ratio of the radius of the spheres to the distance between them be of the magnitude one tenth, the errors are comparable to one one-hundred-millionth.

Properly speaking, the self-induction is not an action at a distance. But the foreign induction is; and is of the second and third orders. The permanent forces of energy are of the second, third, and fourth orders. The temporary forces of energy are of the fifth, sixth, and seventh orders and are called actions of higher order to distinguish them from the others which are all stronger.

Everything is now ready for discussing the mechanics of a system of spheres advancing and pulsating in a fluid — the mechanics which would appear to a person situated in the fluid, and ignorant of its presence or unwilling to take cognizance of the medium as a transmitter of actions which follow the laws of our ordinary mechanics. It is found that for the forces of lower order, that is, for the foreign induction and the permanent forces of energy :

1. The law of inertia holds intact.

2. The second law, that the impulse is in the line of the force and proportional in magnitude to it, also holds.

3. The forces due to distant spheres may be compounded according to the parallelogram law.

4. The forces are independent of the velocity or acceleration of the point of application.

5. The principle of equal action and reaction holds for the forces of energy, and although the action and reaction arising from the induction are not equal and opposite, yet the inequalities influence mainly the motion of the individual spheres and produce no sensible effects on the "molar" motions of systems of spheres in large number, provided only that the distance between the constituent spheres be large in comparison to their radii.

6. The hydrodynamic actions are conservative in nature.

From this it results that to an observer situated in the fluid and made up of spheres as we are made up of atoms the system of mechanics of molar masses of spheres would appear identically to follow the laws of Newton. In reality there would be imperceptible differences, due first to the statement under caption 5 and secondly to the actions of forces of higher orders, which fail to obey most of the six captions stated above.

Of the experimental effects observable upon spheres we have mentioned none directly. The formulas quoted above enable us to infer many facts of more or less obvious analogy, with electric and magnetic phenomena:

1. The self-induction causes a sphere to move more slowly than in free space by the ratio $Q : (Q + \frac{1}{2}q)$.

2. If the external force **f** be chosen so as at each instant to equilibrate the internal force of energy $\mathbf{F}_{en}$, the total motion is due to induction, the induced field is a parallel field, and the sphere moves with the velocity $\frac{3}{2}q : (Q + \frac{1}{2}q)$ of the field.

If by the axis of a doublet or oscillating sphere be meant the direction of the vector kinematic movement of action **S**, then

3. Spheres pulsating in the same phase attract, in opposite phases repel each other.

4. A pulsating sphere and an oscillating sphere whose axis is directed toward the center of the pulsating attract; if the axis is directed in the opposite direction they repel each other.

5. If the axis of the oscillating sphere does not pass through the center of the pulsating sphere, this latter moves in the direction of the axis of the former which at the same time turns its axis toward the pulsating sphere.

6. If two oscillating spheres have parallel axes, attraction takes place in case the directions are the same; repulsion in case they are opposite.

7. If the axes of two oscillating spheres lie in the same line and have the same direction the phenomenon is repulsion; if opposite directions, attraction.

And in general when the axes are arranged arbitrarily the spheres undergo translations and rotations according to the laws which would govern magnets if the signs of the forces were all changed.

To consider the temporary forces of energy, which are of higher order, it is necessary to mask all the actions of lower order. By a glance at the formulas it appears that the actions of lower order vanish if the sphere upon which they are supposed to act is at rest or simply pulsating, and that the temporary actions are increased by increasing the difference in densities between the sphere and the liquid. An investigation shows that:

1. An oscillating sphere attracts a sphere at rest if the sphere be heavier than the liquid, but repulses it if it be lighter. The force is of the seventh order.

2. For a pulsating system the result is the same except that the order of the force is the fifth.

3. Two spheres heavier than the liquid and pulsating with opposite phases repulse each other at long distances and attract at short distances. There is an intermediate position of equilibrium. Spheres lighter than the liquid and pulsating in the same phase attract each other at long distances and repulse at short distances.

The analogies which suggest themselves during the course of the work on hydrodynamic actions at a distance are many. The fact that the motions must be divided into visible and invisible recalls the system of mechanics of Hertz, in which hidden masses and hidden motions are freely introduced to make the system come under the canonical heading. The behavior of the temporary forces calls to mind the laws of molecular actions assumed by theorists for constructing the kinetic theory of gases. All assume a force function which gives attraction at great distances and repulsion at small. The order of the force differs with different authors. The fact that forces of all orders come into play gives a chance to draw analogies with the behavior of matter in regard to gravitation, adhesion and cohesion. But of all the analogies the most fruitful seems to be that which relates to electricity or magnetism. The correspondence seems to be nearly complete. The pulsating sphere corresponds to an electric charge or a magnetic pole; the oscillating sphere, to a doublet or short magnet. The correspondence is, however, negative; where attraction occurs in electricity and magnetism, repulsion is found in hydrodynamics. Why this should be is not known. Perhaps it is only chance, or perhaps it is due to some subtle properties of matter and electricity which hitherto have evaded discovery.

The following table gives in résumé the correspondences which exist between the quantities occurring in hydrodynamics and in electricity or magnetism:

	Electric Medium.	Fluid.
q:	Ponderomotive constant of activity.	Density.
k:	Polarizibility.	Mobility.
$\dot{E}$:	Free pole intensity.	Kinematic intensity of pulsation.
$q\dot{E}$:	True pole intensity.	Dynamic intensity of pulsation.

$\mathfrak{S}$:	True electric or magnetic moment.	Kinematic moment of action.
$\rho\mathfrak{S}$:	Free electric or magnetic moment.	Dynamic moment of action.
$\rho\dot{\mathfrak{S}}$:	Electric or magnetic field intensity.	Hydrodynamic field intensity.
$\dot{\mathfrak{S}}$:	Electric or magnetic polarization.	Velocity.

Having completed our account of the voluminous work under review, it is well to ask what use, what gain accrues to us. In the first place the sole fact that here is a large field of hydrodynamics worked out in detail, analytically and experimentally, is sufficient cause for congratulation. The problems of hydrodynamics which have been carried through to the end are few. The subject and the presentation are often difficult. The theory of the tides, the theory of resistance offered to boats are examples. The theory of C. A. Bjerknes has its difficult points; but so clearly is all explained that the reader does not perceive the burden of the difficulties. At the present state of our knowledge the electric and magnetic analogies seem to be more interesting than valuable. This the authors recognize with their characteristic frankness and no small profit may be obtained by considering the justness with which they estimate and discuss their work.

Another great value and interest in the work lies in its relation to cosmographical speculations. Some years ago one of the presidents of the British Association said in his annual address that since the scientific world, especially the English, had taken such a fancy to devising schemes on which to explain the universe, there would be large interest even if small utility in collecting and comparing the different suggestions of which practically all are of hydrodynamic origin. First is Kelvin's vortex atom and it is doubtless responsible for the rest. Someone pointed out that one vortex anchored at each end to the boundary surface of the universe would answer all purposes. Hicks brought out a "bubble" theory according to which matter was to be hollow spaces in the ether. The development of the theory may be read with great difficulty in McAulay's Utility of Quaternions in Physics. Later Pearson produced an "ether-squirt" theory. An atom, according to him, consists

of a point discharging ether into space with a pulsating rate. The mathematical investigations may be found in the *Cambridge Transactions* and in the *American Journal of Mathematics*. They are not easy to read. C. A. and V. Bjerknes do not pretend to have laid the foundations of the universe. They have treated a broad problem in hydrodynamics and treated it clearly, completely, systematically. To those who would read with any ease the developments of Hicks's and Pearson's theories the work of Bjerknes is a practical necessity; to mathematician and physicist alike it is interesting; and had it appeared twenty years ago when first complete, it would doubtless have attracted much more attention and possessed a much greater influence than now.

EDWIN BIDWELL WILSON.

PARIS,
June, 1903.

SHORTER NOTICES.

Vorlesungen über Geschichte der Trigonometrie. Von DR. A. VON BRAUNMÜHL. Zweiter Teil. Leipzig, B. G. Teubner, 1903. xi + 264 pp.

THE first part of von Braunmühl's History of Trigonometry was reviewed in the BULLETIN, volume 6 (1900), page 404. The second part is fully up to the standard of the first. The completed work will take its place as the fullest and most authoritative history of trigonometry that we have. Cantor's great work, Vorlesungen über Geschichte der Mathematik, comes down only to the year 1758, so that the present history of trigonometry covers nearly one hundred and fifty years of this science which have never been treated before with any degree of thoroughness.

The author begins the second part with the history of logarithms. In this connection he gives an account of John Speidell, the author of the first table of natural logarithms. Hitherto, German writers have overlooked Speidell. De Morgan's interesting account of him, given in the article "Tables" in the English Cyclopœdia, does not seem to have been used by von Braunmühl. He examined the reprint of Speidell's work, the New Logarithmes, that is given in Maseres's Scriptores Logarithmici, volume 6, page 713. Maseres reprinted from the edition of 1628, yet von Braunmühl, unaware of this fact, refers in a footnote, page 26, to a remote source

as the authority for the existence of this edition. It is to be regretted that the Dictionary of National Biography overlooks Speidell. In gleaning information relating to mathematicians and astronomers, it appears that the editors of this great dictionary have not exhibited their usual industry. They have passed by not only Speidell, but also Thomas Streete and John Caswell, all men to whose books von Braunmühl makes repeated reference. It may be worth while, in this connection, to state that mention is made of John Speidell in Aubrey's Brief Lives,* as follows: "Mr. * * * Spiedell: — he taught mathematiques in London, and published a booke in quarto named Spiedel's Geometrical Extractions (London, 163–), which made young men have a love to geometrie."

Von Braunmühl repeatedly points out mistakes in our histories of mathematics. For instance, he calls attention to the error of ascribing to Napier all four of "Napier's analogies." Only two were given by him; the other two were added by Briggs.

The seventeenth century was the period when trigonometric series were first introduced and the nomenclature and notation of trigonometry were in process of rapid development. The author brings to light some interesting notations, suggested by Wallis, Wing, Oughtred, Herigone and others. De Moivre and Cotes recognized the periodicity of the trigonometric functions and discovered the theorems which bear their names. Goniometry, differential trigonometry, and improved methods of computing π were developed in the early part of the eighteenth century. But von Braunmühl brings out strongly the fact that the science was shaped anew by the genius of Leonhard Euler, who advanced analytic trigonometry, treated trigonometric functions as ratios, gave the first derivation of the sine and cosine series without using the calculus, derived as early as 1740 the expressions $\cos v = \frac{1}{2}(e^{v\sqrt{-1}} + e^{-v\sqrt{-1}})$, $\sin v = -\frac{1}{2}\sqrt{-1}\,(e^{v\sqrt{-1}} - e^{-v\sqrt{-1}})$ and such oddities as the expression $\frac{1}{2}(2^{\sqrt{-1}} + 2^{-\sqrt{-1}}) = \frac{10}{16}$, nearly. He invented several methods for the mensuration of the circle, some of them yielding rapid processes of computing π. He also systematized the treatment of spherical trigonometry.

Upon the history of trigonometry during the last century and a half, von Braunmühl has expended much effort. His aim has

* Ed. A. Clark, Oxford, 1898, Volume II, page 231.

been to include the higher developments, which reach into advanced branches of analysis, the theory of functions and of groups. This has led him to discuss the researches of Lambert, Carnot, Legendre, Möbius, Gauss, Cayley and other mathematicians, many of whom are still living.

Von Braunmühl has taken great pains to secure accuracy, and very few errors have been detected by us. On page 107 the author speaks of $i \equiv \sqrt{-1}$ as "das Gausssche i." Probably he does not mean to convey the idea by this that Gauss was the first to use i; this abbreviation was used before Gauss by Euler, as has been pointed out by Beman.* On page 107, line 9, there is an error in algebraic sign in Euler's expression for sin v.

There is evidently an error on page 44 in connection with Thomas Streete, the author of the Astronomia Carolina, 1661. In von Braunmühl's history the name of the astronomer occurs repeatedly and it is everywhere correctly spelled "Streete," except on page 44, where we read "Thomas Street (1626–1696)." Now, we are going to show that there were two distinct individuals, a Thomas Streete, the astronomer and a Thomos Street, the judge. The two were contemporaries, but the dates given by von Braunmühl apply to the judge, whose life is sketched quite fully in the Dictionary of National Biography. Wolf says in his Geschichte der Astronomie: "Streete designated himself on the title page [of his Astronomia Carolina] a 'Student in Astronomy and Mathematics'—was, therefore, in 1661, evidently still quite a young man. Otherwise nothing relating to him appears to have been handed down." Because of this paucity of biographical detail there is danger that an error in dates and in the identity of the man may not be easily detected and, therefore, may be repeated by other writers. We have been able to gather the following details relating to Thomas Streete, the astronomer.

1. Streete is the author of the Astronomia Carolina, 1661, and also of works bearing the following titles: Examen Examinatum, or Wing's Examination of Astronomia Carolina examined * * * with a castigation of the Envy and Ignorance of Vincent Wing, by Thomas Streete, Student in Astronomy and Mathematicks * * *, London, 1667; The Genuine Use and Effects of the Gunne, by Robert Anderson, with Tables of Projection by Thomas Streete, London, 1674; A Compleat Eph-

* BULLETIN, volume 4 (1898), page 274.

emeries for the Year of Christ 1684, by Thomas Streete, the author of Astronomia Carolina and Teacher of the Mathematics, London, 1684. At the end of his Ephemeries he says: "Touching my further progress in the work of my intended large astronomical tables, the obstacle is no other than my being as yet necessitated to labour upon other things for a present livelyhood * * * ."

2. In the appendix to his Astronomia Carolina he states that he took observations of lunar eclipses with Mr. Moxon and Mr. R. Anderson in Cornehill. In the preface to the edition of the Astronomia Carolina, brought out by Halley in 1710, it is stated "that our author, before his death, had attempted to alter some things in his tables," but had left "very little of the work done."

3. Savérien in his Dictionnaire universel de mathématique, Paris, 1753, article "Quartier Anglois" says that after Hooke's invention of a quartier anglois (back-staff): "M. Stréet, auteur de l'Astronomie Caroline, inventa ensuite un autre Quartier Anglois, garni de deux plans au travers desquels il regardoit un object directement, & il trouvoit l'autre par la simple reflexion d'un morceau de miroir."*

4. In Aubrey's Brief Lives (a source of information generally overlooked by the historians of mathematics) there are a number of details about Thomas Streete. According to Aubrey, Streete was born in Ireland, at Castle Lyons, March 5, 1621. "He dyed in Chanon-row (vulgarly Channel-rowe) at Westminster, the 17th of August, 1689, and is buried in the church yard of the new chapell there towards the east window of the chancel, scilicet, within twenty or 30 foot of the wall. Hee made this following epitaph himself:—

> 'Here lies the earth of one that thought some good,
> Although too few him rightly understood:
> Above the starres his heightned mind did flye,
> His hapier spirit into Eternity.'"

His personality is described thus: "He was of a rough and cholerique humour. Discoursing with Prince Rupert, his highnesse affirmed something that was not according to art; sayd Mr. Street, 'whoever affirmes that is no mathematician.' So they would point at him afterwards at court and say, 'There's the man that huff't prince Rupert.'"

* See also Savérien's Histoire des Progrès de l'Esprit Humain dans les Sciences, Paris, 1776, pp. 160, 161.

"He hath left with his widowe (who lives in Warwick lane at the signe of the * * *) an absolute piece of Trigonometrie, plain and spherical, in MS., more perfect than ever was yet donne, and more cleare and demonstrated." Aubrey says that Streete published almanacs, "for about three yeares, dedicated to Elias Ashmole, esquire: but was not encouraged for his great paines.—He was one of Mr. Ashmole's clarkes in the Excise office, which was his chiefest lively-hood."

FLORIAN CAJORI.

Réflexions sur la Puissance Motrice du Feu et sur les Machines propres à Développer cette Puissance. Par SADI CARNOT. Réimpression facsimile conforme à l'édition originale de 1824. Paris, A. Hermann, 1903.

Eighty years ago Carnot published his short book which, taken with the work of Joule and Mayer a few years later, forms the basis of our present theories of heat. At that time M. Bachelier, the predecessor of Gauthier-Villars, probably found the work unpopular. Of late years the demand for the original has far exceeded the supply. It is with the purpose of meeting this demand as satisfactorily as possible and at a reasonable price that the work has been reprinted. To render the reproduction exact, even to the misspelling, the photographic method has been employed. This, with the fact that paper to imitate the old has been selected, makes the new edition practically as valuable as the original even from the standpoint of the bibliophile. Not long ago M. Hermann republished in a similar manner the Théorie des Nombres of Legendre, and more recently the Mathematical Papers of Green. Let us hope that as the great classics of scientific literature become rare, someone may always be found ready to reprint them. Such a procedure is often more desirable than translation or editing as in Ostwald's or Appleton's series of scientific classics. It puts at our disposal an exact reproduction of the original.

E. B. WILSON.

NOTES.

THE Christmas meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at St. Louis, in conjunction with the meeting of Section A of the American association for the advancement of science, on Thursday and Friday, December 31 and January 1. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor Thomas F. Holgate, 617 Hamline Street, Evanston, Ill., not later than December 12.

THE closing (October) number of volume 4 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the subgroups of order a power of p in the quaternary abelian group in the Galois field of order p^n," by L. E. DICKSON; "On the order of linear homogeneous groups," by H. F. BLICHFELDT; "Non-abelian groups in which every subgroup is abelian," by G. A. MILLER and H. C. MORENO; "On nilpotent algebras," by J. B. SHAW; "On solutions of differential equations which possess an oscillation theorem," by HELEN A. MERRILL; "On the reducibility of linear groups," by L. E. DICKSON; "Semireducible hypercomplex number systems," by S. EPSTEEN; "A symbolic treatment of the theory of invariants of quadratic differential quantics of n variables," by H. MASCHKE; "Congruences of curves," by L. P. EISENHART; "Similar conics through three points," by T. J. I'A. BROMWICH.

THE opening (October) number of volume 5 of the *Annals of Mathematics* contains: "The mathematical theory of the top, I," by A. G. GREENHILL; "On quadratic forms," by PAUL SAUREL; "A generalized conception of area: applications to collineations in the plane," by E. B. WILSON; "Lines of curvature on minimum developables," by F. S. WOODS.

THE Association of mathematical teachers in New England met on Saturday, November 14, at 11 o'clock at The Browne and Nichols School, 20 Garden Street, Cambridge, Mass. Mr. G. W. EVANS, of the English High School, Boston, read a paper on "Graphical representation in arithmetic and algebra," and Dr. E. V. HUNTINGTON, of Harvard University, gave a short account of Holzmüller's Stereometry.

THE first regular meeting of the Association of teachers of mathematics in the Middle States and Maryland will be held at Columbia University on Saturday, November 28, 1903. The following programme is announced: Address of welcome, by President BUTLER; "The laboratory method of teaching mathematics," by HARRY ENGLISH; "The syllabus for geometry in the grades recently adopted for the New York schools," by I. N. FAILOR; "Suggested topics for scientific investigation by this association," by ARTHUR SCHULTZE; "The report of the committee of the AMERICAN MATHEMATICAL SOCIETY on college entrance requirements," by J. L. PATTERSON. An exhibition of models, apparatus, rare books, and autographs, of interest to teachers of mathematics, will also be held during the meeting.

THE international congress of arts and sciences, to be held in connection with the St. Louis Exposition, will extend through the week beginning September 19, 1904. Sectional meetings, either separately or in conjunction with other congresses or societies, will be continued in the following week, so far as may be found desirable. The administrative board has appointed Professor SIMON NEWCOMB president of the congress. The preliminary programme discloses an authoritative classification of the arts and sciences into seven divisions, twenty-six departments, and one hundred and thirty-one sections. Mathematics is grouped with philosophy as a normative science, and is divided into three sections: Algebra and analysis, Geometry, and Applied Mathematics. A large number of eminent scientists from abroad have been invited to participate in the congress. In each of the sections two addresses will be delivered, one on the relations to other sciences and one on the problems of to-day. An opening address will also be delivered by the president of the congress on its relations to scientific progress. The entire series of addresses, 322 in all, will be published by the exposition as a permanent memorial.

As the international congress of mathematicians at Heidelberg closes on August 13, it will be quite practicable for those desiring to do so to attend both meetings.

AN inquiry recently made in nine American universities as to the number of students taking certain advanced courses offered last year in the institutions considered, shows the following: Theory of functions of a complex variable, 67; Projective

geometry, 94; Theory of invariants, 26; Theory of groups, 45; Modern advanced theory of equations, 46; Theory of elliptic functions, 11. The numbers are suggestive of the growth of advanced courses in this country, although the statistics are unsatisfactory from the fact that certain other equally advanced courses are omitted, certain of those named are occasionally treated as elementary, and several institutions give the substance of these theories in other courses.

CAMBRIDGE UNIVERSITY. — Advanced mathematical courses for the current academic year are announced as follows:

Michaelmas term, 1903. — By Professor A. R. FORSYTH: Partial differential equations, three hours; Calculus of variations, two hours. — By Professor G. H. DARWIN: Theory of potential and attractions, three hours. — By Professor Sir R. S. BALL: Planetary theory, three hours. — By Professor J. LARMOR: Electrodynamics. — By Professor J. J. THOMSON: Properties of matter, two hours; Electricity and matter, two hours. — By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours. — By Mr. H. W. RICHMOND: Analytic geometry of curves, three hours. — By Dr. H. F. BAKER: Elementary theory of functions, three hours; Solid geometry, three hours. By Mr. H. M. MACDONALD: Waves (especially waves of light), three hours. — By Mr. G. B. MATHEWS: Theory of numbers, three hours. — By Mr. A. N. WHITEHEAD: Application of symbolic logic to Cantor's theory of aggregates, three hours. — By Mr. A. BERRY: Elliptic functions, Bessel functions and Fourier series, three hours. — By Mr. A. MONRO: Hydrodynamics and sound, three hours. — By Mr. E. T. WHITTAKER: The differential equations of applied mathematics, two hours. — By Mr. J. H. GRACE: Invariants and geometric applications, three hours.

Lent term, 1904. — By Professor A. R. FORSYTH: Partial differential equations (continued), three hours. — By Professor G. H. DARWIN: Figure of the earth and precession, three hours. — By Professor Sir R. S. BALL: Application of geometry to dynamics, three hours. — By Professor J. LARMOR: Electrodynamics with optical applications, three hours; Elementary mathematical physics, three hours. — By Professor J. J. THOMSON: Electricity and magnetism, three hours; Discharge of electricity through gases, two hours. — By Dr. E. W. HOBSON: Sound and vibrations, three hours. — By Mr. H. W. RICHMOND: Analytic geometry of three dimensions, projective

properties, three hours. — By Dr. H. F. BAKER: Theory of functions, three hours; Analysis. — By Mr. H. M. MACDONALD: Hydrodynamics. — By Mr. R. A. HERMAN: Hydromechanics, six hours (two courses). — By Mr. G. B. MATHEWS: Theory of numbers (continued), three hours. — By Mr. A. BERRY: Elliptic functions. — By Mr. G. T. BENNETT: Linear and quadratic complexes, three hours. — By Mr. E. T. WHITTAKER: General dynamics. — By Mr. J. H. GRACE: Invariants and geometric applications (continued), three hours.

Easter term, 1904. — By Professor J. LARMOR: The theory of gases and molecular statistics of energy, three hours. — By Professor J. J. THOMSON: Electricity and magnetism, three hours. — By Dr. H. F. BAKER: Theory of groups, three hours; Analysis (continued), three hours. — By Mr. W. L. MOLLISON: Theory of potential and electrostatics, three hours. — By Mr. A. N. WHITEHEAD: Non-euclidean geometry. — By Mr. E. T. WHITTAKER: The nebular hypothesis and related investigations, two hours.

Long vacation, 1904. — By Professor Sir R. S. BALL: Perturbation of cometary orbits, three hours. — By Mr. H. W. RICHMOND: Algebraic curves in connection with automorphic functions. — By Mr. W. M. COATES: Electricity and magnetism, three hours. — By Mr. J. G. LEATHEM: Physical optics, three hours.—By Mr. R. W. H. T. HUDSON: Kummer's quartic surface.

THE publication of an *Annuaire internationale des sociétés savantes*, edited by H. DELAUNY, is announced. The *Annuaire* notices over 5,000 learned societies.

THE second series of the Decennial Publications of the University of Chicago will include a volume, soon to appear, entitled Lectures on the Calculus of Variations, by OSKAR BOLZA. This is in substance a reproduction in considerably extended form of the lectures delivered by the author at the colloquium held in connection with the summer meeting of the AMERICAN MATHEMATICAL SOCIETY, at Ithaca, N. Y., in August, 1901.

Volume 8 of the first series of the Decennial Publications, which appeared last August, contained a paper, "On certain rigorous methods of treating problems in celestial mechanics," by F. R. MOULTON. Volume 9, devoted to mathematics, chem-

istry, physics, and geology, which also appeared in August, contains: "Concerning the geodesic curvature and the isoperimetric problem on a given surface" and "Proof of the sufficiency of Jacobi's condition of a permanent sign of the second variation in the so-called isoperimetric problems," by OSKAR BOLZA; "Ternary orthogonal groups in a general field" and "The groups defined for a general field by the rotation group," by L. E. DICKSON; "Invariants and covariants of quadratic differential quantics of n variables," by H. MASCHKE; "The subgroups of the generalized finite modular group," by E. H. MOORE.

The articles which appeared in the first series are also issued separately as reprints, with heavy paper covers.

THE Prince Jablonowski society of Leipsic has awarded to Professor ERNST NEUMANN, of Breslau, its prize of 1,000 marks for an elaboration of Poincaré's memoir "On the method of Neumann and Dirichlet's problem," published in the *Acta Mathematica*, 1896.

The subjects proposed by the society for the years 1903 and 1904 have been noted in recent numbers of the BULLETIN.

THE Reale Accademia dei Lincei of Rome has elected Professor D. HILBERT, of Göttingen, to a foreign fellowship. The academy has deferred the award of the royal prize for mathematics for the current year.

DR. A. KORN has been promoted to a professorship of mathematics at the University of Munich. Dr. O. KRIGAR-MENZEL has been appointed to succeed the late Professor J. Weingarten in the Charlottenburg technical high school; O. SINTZOFF has been appointed to a professorship of mathematics in the University of Charkoff.

PROFESSOR W. WIRTINGER, of Innsbuck, has been called to the University of Vienna.

THE American academy of arts and sciences has elected, as foreign honorary members, Professors E. PICARD, of Paris, and J. LARMOR, of Cambridge.

PROFESSOR SIMON NEWCOMB has been made one of the vice-presidents of section A (mathematics and physics) of the British association for the advancement of science. The next meeting of the association will be held at Cambridge, beginning August 17, 1904.

THE college entrance examination board has appointed as examiners in mathematics for 1904, Professor F. N. COLE, Professor T. C. ESTY, and Dr. ARTHUR SCHULZE.

DR. H. E. HAWKES has been promoted to an assistant professorship of mathematics at Yale University.

MR. C. R. BURGER has been appointed assistant professor of mathematics at the Colorado School of Mines.

MR. A. C. MINEAR has been appointed professor of mathematics in the University of Southern California, to succeed Professor Paul Arnold.

THE following appointments have also been recently announced: Mr. C. GILMAN, as instructor in mathematics and surveying in Harvard University; Mr. F. C. EDMINSTER, as assistant in mathematics in Cornell University.

PROFESSOR J. F. DOWNEY, head of the department of mathematics in the University of Minnesota, has been appointed to the newly created office of dean of the faculty of science, literature and arts.

THE death is announced of Professor RUDOLF LIPSCHITZ, of the University of Bonn. Professor Lipschitz only recently celebrated the fiftieth anniversary of his doctorate.

THE following catalogues of second-hand mathematical works have recently been received: W. Heffer & Sons, 4, Petty Cury, Cambridge, England, 1485 numbers, including the library of the late Professor Stokes; Gustav Fock, 7 Schlossgasse, Leipzig, catalogue no. 230, 5419 numbers; A. Geering, Bäumleingasse 10, Basel, catalogue no. 292, 252 works on mathematics; W. Junk, 22 Rathenower Strasse, Berlin, catalogue no. 23, 40 pages.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BEER (K.). Ueber die Möglichkeit einer algebraischen Teilung der Lemniskatenperipherie. (Progr.) Kremsier, 1903. 8vo. 10 pp., 1 plate.

BIANCHI (L.). Lezioni di geometria differenziale. 2a edizione, riveduta ed aumentata. Vol. II. Pisa, 1903. 8vo. 604 pp. Fr. 22.50

CAUCHY (A.). Oeuvres complètes d'Augustin Cauchy. Publiées sous la direction scientifique de l'Académie des sciences et sous les auspices de M. le ministre de l'instruction publique. 2e série. Vol. V. Paris, Gauthier-Villars, 1903. 4to. 532 pp.

CHABASOFF (G.). Arithmetische Untersuchungen über Irreduktibilität. (Diss.) Heidelberg, 1902. 8vo. 67 pp. M. 2.00

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. III: Geometrie. Herausgegeben von W. F. Meyer. Teil 3, Heft 2–3: Scheffers (G.), Besondere transcendente Kurven; Lilienthal (R. von), Besondere Flächen; Voss (A.), Abbildung und Abwickelung zweier Flächen auf einander. Leipzig, Teubner, 1903. 8vo. Pp. 185–440.

GRACE (J. H.) and ROSENBERG (F.). Co-ordinate geometry: the conic. London, Clive, 1903. 8vo. Cloth. 3s. 6d.

HANCOCK (H.). Mémoire sur les systèmes modulaires de Kronecker. (Thèse.) Paris, Gauthier-Villars, 1901. 4to. 115 pp.

——. Remarks on Kronecker's modular systems. Paris, Gauthier-Villars, 1902. 8vo. 33 pp.

JAHRBUCH über die Fortschritte der Mathematik. Begründet von C. Ohrtmann, herausgegeben von E. Lampe und G. Wallenberg. Vol. 32: Jahrgang 1901. Heft 1. Berlin, Reimer, 1903. 8vo. Pp. 1–480. M. 15.00

KOMMERELL (V. UND K.). Allgemeine Theorie der Raumkurven und Flächen. Vol. II. Leipzig, Göschen, 1903. 12mo. 4 + 212 pp. Cloth. (Sammlung Schubert, No. XLIV.) M. 5.80

LAMPE (E.). See JAHRBUCH.

LILIENTHAL (R. VON). See ENCYKLOPÄDIE.

MEYER (W. F.). See ENCYKLOPÄDIE.

ROSENBERG (F.). See GRACE (J. H.).

SCHEFFERS (G.). See ENCYKLOPÄDIE.

SCHUBERT (H.). Niedere Analysis. Teil 2: Funktionen; Potenzreihen; Gleichungen. Leipzig, Göschen, 1903. 12mo. 5 + 215 pp. Cloth. (Sammlung Schubert, No. XLV.) M. 3.80

SISAM (C. H.). On directrix curves of quintic scrolls. (*Bulletin of the American Mathematical Society*, Vol. 10, pp. 32–34.) New York, 1903.

TROPFKE (J.). Geschichte der Elementar-Mathematik in systematischer Darstellung. Vol. II: Geometrie; Logarithmen; ebene Trigonometrie; Sphärik und sphärische Trigonometrie; Reihen; Zinseszinsrechnung; Kombinatorik und Wahrscheinlichkeitsrechnung; Kettenbrüche; Stereometrie; analytische Geometrie; Kegelschnitte; Maxima und Minima. Leipzig, Veit & Co., 1903. 8vo. 8 + 496 pp. M. 12.00

VOSS (A.). See ENCYCKLOPÄDIE.

WALLENBERG (G.). See JAHRBUCH.

WEBER (H.) und WELLSTEIN (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrende und Studierende. Vol. I: Elementare Algebra und Analysis, bearbeitet von H. Weber. Leipzig, Teubner, 1903. 8vo. 14 + 447 pp. Cloth. M. 8.00

WELLSTEIN (J.). See WEBER (H.).

II. ELEMENTARY MATHEMATICS.

ALLCOCK (C. H.). Theoretical geometry for beginners. Part 2. London and New York, Macmillan, 1903. 12mo. 4 + 123 pp. Cloth. $0.35

BAKER (W. M.) and BOURNE (A. A.). Elementary geometry. Books 2, 3. London, Bell, 1903. 8vo. 160 pp. Cloth. 1s. 6d.

——. Elementary geometry. Books 2–4. London, Bell, 1903. 8vo. 200 pp. Cloth. 2s. 6d.

BARNARD (S.) and CHILD (J. M.). A new geometry for schools. New York, Macmillan, 1903. 12mo. 26 + 514 pp. Cloth. $1.10

BIEL (B.) Mathematische Aufgaben für die höheren Lehranstalten, unter möglichster Berücksichtigung der Anwendungen, wie überhaupt der Verknüpfung der Mathematik mit anderen Gebieten zusammengestellt. Ausgabe für Gymnasien. Teil 1: Die Unterstufe. Leipzig, Freytag, 1903. 8vo. 6 + 161 pp. Cloth. M. 2.50

BORJES Y ALEGRE (E.). Elementos de trigonometría rectilínea. Toledo, Serrano, 1902. 8vo. 82 pp.

BORK (H.). Mathematische Hauptsätze. Ausgabe für Gymnasien. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil 2: Pensum der Oberstufe (bis zur Reifeprüfung). 3te, durchgesehene, teilweise umgearbeitete und den preussischen Lehrplänen von 1901 angepasste Auflage. Leipzig, Dürr, 1903. 8vo. 12 + 388 pp. Cloth. M. 3.60

BOURNE (A. A.). See BAKER (W. M.).

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales (I, division centésimale; II, division sexagésimale), conformes à l'arrêté ministériel du 3 août 1901, à l'usage des candidats au baccalauréat et aux Ecoles polytechnique et de Saint-Cyr. 3e édition, revue et corrigée. Paris, Hachette, 1903. 8vo. 176 pp. Fr. 2.50

BRIGGS (W.). Second stage mathematics. 3d edition. (Organized Science Series.) London, Clive, 1903. 8vo. 474 pp. Cloth. 3s. 6d.

CHILD (J. M.). See BARNARD (S.).

DEAKIN (R.). Euclid, books 1–4. Introductory course of drawing and measurement; and problems on practical geometry. 2d edition. 3d imprint. (University Tutorial Series.) London, Clive, 1903. 8vo. 344 pp. Cloth. 2s. 6d.

DELSOL (E.). Principes de géométrie. Paris, Naud, 1903. 8vo. 98 pp.

EUCLID. See DEAKIN (R.).

FINN (S. W.). Junior algebra examination papers. London, Methuen, 1903. 12mo. Limp leather. 1s.

GODFREY (C.) and SIDDONS (A. W.). Elementary geometry, practical and theoretical. Vol. 1: experimental course and books 1 and 2. Vol. 2: books 3 and 4. London, Clay, 1903. 8vo. Cloth. 3s. 6d.

GREILACH (S.). Zur Quadratur des Kreises. (Progr.) St. Paul, 1903. 8vo. 42 pp.

HALL (H. S.) and STEVENS (F. H.). A school geometry, parts 1, 2 and 3. Part 1, lines and angles, rectilineal figures; part 2, areas of rectilineal figures; part 3, circles; containing the substance of Euclid, book 1, book 3, 1–34, and part of book 4. New York, Macmillan, 1903. 12mo. 11 + 214 pp. Cloth. $0.60

HÖHR (D.). Lehrbuch der Arithmetik für Untergymnasien. Mit Anhang für Realschulen. Teil 1: für die 1ste und 2te Klasse. 3te Auflage. Nagyszeben, Krafft, 1903. 8vo. 8 + 160 pp. Cloth. M. 1.79

HUSBAND (J.). See MORRIS (I. H.).

IBOR GUARDIA (A.). Elementos de aritmética y algebra, espuestos simultáneamente, en el que se estudian á la vez dichas dos ramos principales de la matemática, consiguiendo por este medio adelantar mucho en su estudio empleando para ello poco tiempo. Madrid, 1902. 8vo. 250 pp. Fr. 8.00

——. Geometría y trigonometría, espuestas por un nuevo método (fusión de las geometrías y sus proyecciones) que proporciona á los que le siguen economía de trabajo, reducción del gran numero de teoremas geométricos á su mitad, y por la sencillez de sus demostraciones, aprende mucho en muy poco tiempo. Madrid, 1903. 8vo. 300 pp. Fr. 10.00

——. Ejercicios y problemas de aritmética y algebra, con sus corrispondientes soluciones, dispuestas para los alumnos de los institutos generales y técnicos, que resuelven ellos mismos en la parte derecha de estos ejercicios. Madrid, 1903. 64 pp. Fr. 1.50

——. Ejercicios y problemas de geometría, trigonometría y cosmografía, con sus corrispondientes soluciones, dispuestos para los alumnos des los institutos generales y técnicos, que resuelven ellos mismos en la parte derecha de estos ejercicios. Madrid, 1903. 8vo. 64 pp. Fr. 1.50

JACQUET (E.) et LACLEF (A.). Cours de géométrie théoretique et pratique, avec de nombreux exercices, problèmes, applications, etc., à l'usage des lycées et collèges (premier et deuxième cycles, sections A et B). Paris, Nathan, 1904. 16mo. 519 pp.

KRUG. Die niedere Analysis auf der Unterrichtsstufe des Realgymnasiums. (Progr.) Stuttgart, 1903. 4to. 43 pp.

KÜBLER (J.). Die Proportion des goldenen Schnitts als das geometrische Ziel der stetigen Entwicklung und die daraus hervorgehende Fünfgestalt mit ihrer durchgreifenden Fünfgliederung. Leipzig, Teubner, 1903. 8vo. 36 pp., 4 plates. M. 1.60

LACLEF (A.). See JACQUET (E.).

LIGHTFOOT (J.). Elementary and intermediate algebra. Exercises, answers. London, Ralph, Holland, 1903. 8vo. Cloth. 4s. 6d.

MAYER (J. E.). Das mathematische Pensum des Primaners. Ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien, sowie zum Selbstunterricht. Heft 6 und 7: Sätze und Aufgaben aus der Geometrie: Potenz, Satz des Menelaos, des Ceva, harmonische Punkte und Strahlen, Satz des Pascal, des Brianchon, Pol und Polare etc. etc.; Grundzüge der geometrischen Projektionslehre. Freiburg i. B., Lorenz, 1903. 8vo. 80 pp. M. 2.00

——. Hefte 11 und 12: Kombinatorik (Permutation, Kombination, Variation); Wahrscheinlichkeitsrechnung, Versicherungsrechnung; imaginäre Zahlen (Moivre'scher Satz); Maxima und Minima. Freiburg i. B., Lorenz, 1903. 8vo. 77 pp. M. 2.00

MORRIS (I. H.) and HUSBAND (J.). Practical plane and solid geometry; adapted to the requirements of the revised syllabus of the Board of Education. New edition. New York, Longmans, 1903. 12mo. 4 + 254 pp. Cloth. $0.80

MÜLLER (H.). See ZWERGER (M.).

NATH (M.). See BORK (H.).

PIONCHON (J.). Evaluation numérique des grandeurs géométriques. Paris, 1903. 8vo. Fr. 3.50

RATINET (A.). See BOUVART (C.).

SANDERS (A.). Elements of plane and solid geometry. New York, American Book Co., 1903. 12mo. 384 pp. Half leather. $1.25

——. Elements of solid geometry: being books 6–9 of "Elements of plane and solid geometry." New York, American Book Co., 1903. 12mo. Pp. 248–384. Cloth. $0.75

SCHUBERT (H.). Elementare Berechnung der Logarithmen; eine Ergänzung der Arithmetik-Bücher. Leipzig, Göschen, 1903. 8vo. 87 pp. M. 1.60

SIDDONS (A. W.). See GODFREY (C.).

SIMON (P.). Traité de géométrie élémentaire, à l'usage des élèves de l'enseignement secondaire (premier et deuxième cycles), suivi de compléments à l'usage des candidats aux écoles du gouvernement. Ouvrage conforme aux programmes officiels du 31 mai 1902. Paris, Belin, 1903. 8vo. 189 pp.

SKORCZYK (F.). Leitfaden der Geometrie für Präparanden-Anstalten und Seminare. Teil 2: Ebene Trigonometrie und Stereometrie. Halle, Schroedel, 1903. 8vo. 8 + 150 pp. M. 1.60

STEVENS (F. H.). See HALL (H. S.).

VILLETTE (T.). Eléments de géométrie pratique et de dessin linéaire. (Cours moyen et supérieur) 6e édition. Paris, Larousse, 1903. 8vo. 2 + 174 pp.

WEBER (H.) und WELLSTEIN (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrende und Studierende. Vol. I: Elementare Algebra und Analysis, bearbeitet von H. Weber. Leipzig, Teubner, 1903. 8vo. 14 + 447 pp. Cloth. M. 8.00

WELLSTEIN (J.). See WEBER (H.).

ZWERGER (M.). Leitfaden zum Unterrichte in der elementaren Mathematik, mit einer Sammlung von Aufgaben. 12te Auflage des Leitfadens der Mathematik von H. Müller. 1ste, 3te und 4te Abteilung. München, Lindauer, 1903. 8vo. (I. Arithmetik. 8 + 246 pp. III. Trigonometrie. 7 + 55 pp. IV. Räumliche Geometrie. 7 + 85 pp.) M. 4.20

III. APPLIED MATHEMATICS.

ABBOT (C. G.). The relation of the sun-spot cycle to meteorology. 8vo. 8 pp. (*Monthly Weather Review*, April, 1902.)

ABBOTT (H. L.). Climatology of the Isthmus of Panama. Mean barometric pressure at sea level on the American Isthmus. 8vo. 24 pp. (*Monthly Weather Review*, March, 1903.)

ALLAN (S. J.). Radioactivity of freshly fallen snow. 8vo. 3 pp. (*Monthly Weather Review*, December, 1902.)

ANTOMARI (X.). Cours de géométrie descriptive, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, aux Ecoles centrale des arts et manufactures, des ponts et chaussées et des mines de Paris et de Saint-Etienne. 2e édition, conforme au nouveau programme d'admission à l'Ecole polytechnique. Paris, Nony, 1903. 8vo. 621 pp.

ARNOLD (E.). Die Gleichstrommaschine. Theorie, Konstruktion, Berechnung, Untersuchung und Arbeitsweise derselben. Vol. II: Konstruktion, Berechnung, Untersuchung und Arbeitsweise der Gleichstrommaschine. Berlin, Springer, 1903. 8vo. 15 + 655 pp., 11 plates. Cloth. M. 18.00

BARNETT (S. J.). Elements of electromagnetic theory. New York, Macmillan, 1903. 12mo. 480 pp. Cloth. $3.00

BUCKINGHAM (E.). Note on the radiation formulas and the principles of thermometry. 8vo. 6 pp. (*Monthly Weather Review*, April, 1903.)

COLLIGNON (E.). Traité de mécanique. 2e partie: statique: 4e édition. Paris, Hachette, 1903. 8vo. 670 pp. Fr. 7.50

DAVIS (H. N.) and KIMBALL (H. H.). Observations of solar radiation with the Angström pyrheliometer. 4to. 19 pp. (*Monthly Weather Review*, June and July, 1903.)

DAVIS (T. H.). Annual wind resultants. 4to. 4 pp. (*Monthly Weather Review*, November, 1902.)

DINTZL (E.). Ueber den Satz vom Kräfteparallelogramm. (Progr.) Wien, 1903. 8vo. 9 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. IV: Mechanik. Herausgegeben von F. Klein. Teil 1, Heft 3: JUNG (G.), Geometrie der Massen; Henneberg (L.), Die graphische Statik der starren Körper. Leipzig, Teubner, 1903. 8vo. Pp. 279–434.

FUHRMANN (A.). Anwendungen der Infinitesimalrechnung in den Naturwissenschaften, im Hochbau und in der Technik. Lehrbuch und Aufgabensammlung. In sechs Teilen, von denen jeder ein selbständiges Ganzes bildet. Teil IV: Bauwissenschaftliche Anwendungen der Integralrechnung. Lehrbuch, Aufgabensammlung und Literaturnachweis. Berlin, Ernst, 1903. 8vo. 14 + 292 pp. M. 9.00

GLAZEBROOK (R. T.). Electricity and magnetism: an elementary text-book, theoretical and practical. New York, Macmillan, 1903. 12mo. 8 + 440 pp. Cloth. $2.00

HALL (H. R.). Governors and governing mechanism. London, Technical Publ. Co., 1903. 8vo. 128 pp. Cloth. 2s. 6d.

HEFTY (J.). See STEINMETZ (C. P.).

HENNEBERG (L.). See ENCYKLOPÄDIE.

HEUBACH (J.). Der Drehstrommotor. Ein Handbuch für Studium und Praxis. Berlin, Springer, 1903. 8vo. 18 + 356 pp. Cloth. M. 10.00

HOSKINS (L. M.). Theoretical mechanics: an elementary text-book. 2d edition. Standford University, Cal., Stanford University Bookstore, 1903. 8vo. Cloth. $3.00

JAMIESON (A.). Text-book of applied mechanics and mechanical engineering. Vol. 1. Examination questions. 4th edition. London, Griffin, 1903. 8vo. 564 pp. Cloth. 8s. 6d.

JUNG (G.). See ENCYKLOPÄDIE.

KIMBALL (H. H.). See DAVIS (H. N.).

KLEIN (F.). See ENCYKLOPÄDIE.

MICHEL (C.). Cours de mécanique à l'usage des candidats à l'Ecole polytechnique. Paris. 8vo. 138 pp. Fr. 3.00

MILLIKAN (R. A.). Mechanics, molecular physics, and heat: a twelve weeks' college course. Boston, Ginn, 1903. 8vo. 242 pp. Cloth. $1.50

MOOSER (J.). Theorie der Entstehung des Sonnensystems. Eine mathematische Behandlung der Kant-Laplace'schen Nebularhypothese. St. Gallen, Fehr, 1903. 8vo. 30 pp. M. 1.20

NEUREITER (F.). Die Verteilung der elektrischen Energie. 2te, neu verfasste Auflage. Leipzig, Leiner, 1903. 8vo. 10 + 276 pp. M. 9.00

OGG (A.). See PLANCK (M.).

PELLAT (H.). Cours d'électricité. Tome 2: électrodynamique; magnétisme; induction; mesures électromagnétiques. Paris, Gauthier-Villars, 1903. 8vo. 558 pp. Fr. 18.00

PLANCK (M.). Treatise on thermodynamics. Translated, with the author's sanction, by A. Ogg. London and New York, Longmans, 1903. 8vo. 12 + 272 pp. $3.00

PLUMANDON (J. R.). Cannon and hail. 8vo. 8pp. (*Monthly Weather Review*, Annual Summary, 1902.)

REDONDO Y TEJERO (H.). Cálculo grafico de los elementos que determinan la precisión alcanzada en la medida de una magnitud y la eficacia de un shrapnel. Madrid, 1903. 4to. 29 pp., 5 plates. Fr. 5.00

SCHUBERT (T.). Die Entstehung der Planeten-, Sonnen- und Doppelsternsysteme und aller Bewegungen in denselben aus den Elementen ihrer Bahnlinien nachgewiesen. Bunzlau, Kreuschmer, 1903. 8vo. 5 + 83 pp., 14 plates. M. 3.00

SCIENCE (intermediate) applied mathematics papers. Questions set at the University of London, 1887–1903. (University Tutorial Series.) London, Clive, 1903. 8vo. 58 pp. 2s. 6d.

SHAW (W. N.). On curves representing the paths of air in a special type of traveling storm. 8vo. 6 pp. (*Monthly Weather Review*, July, 1903.)

SONDERICKER (J.). Graphic statics with applications to trusses, beams, and arches. New York, Wiley, 1903. 8vo. 8 + 137 pp. Cloth. $2.00

STEBBING (F. C.). Navigation and nautical astronomy. London, Macmillan, 1903. 8vo. 352 pp. Cloth. 8s. 6d.

STEINMETZ (C. P.). Theoretische Grundlagen der Starkstrom-Technik. Uebersetzt von J. Hefty. Braunschweig, Vieweg, 1903. 8vo. 11 + 331 pp. M. 9.00

TETMAJER (L. VON). Die Gesetze der Knickungs- und der zusammengesetzten Druckfestigkeit der technisch wichtigsten Baustoffe. 3te, vervollständigte Auflage. Wien, Deuticke, 1903. 8vo. 8 + 211 pp., 6 plates. M. 8.00

THIESSEN (A. H.). An explanation of wireless telegraphy. 8vo. 17 pp. (*Monthly Weather Review*, December, 1902.)

THOMSON (J. J.). Conduction of electricity through gases. London, Clay, 1903. 8vo. 574 pp. Cloth. 16s.

WOLFER (A.). Revision of Wolf's sun-spot relative numbers. 4to. 6 pp. (*Monthly Weather Review*, April, 1902.)